Right Gaussian rings and related topics

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Michał Ziembowski)

To Beata, Joanna and Mateusz

Abstract

Prüfer domains are commutative domains in which every non-zero finitely generated ideal is invertible. Since such domains play a central role in multiplicative ideal theory, any equivalent condition to the Prüfer domain notion is of great interest. It occurs that the class of Prüfer domains is equivalent to other classes which are investigated in theory of commutative rings (see [22]). Namely, for commutative rings the following classes are equivalent:

(1) Semihereditary domains.

(2) Domains which have weak dimension less or equal to one.

(3) Distributive domains.

(4) Gaussian domains.

(5) Prüfer domains.

Many authors have studied so called Prüfer rings which are a generalization of notion of Prüfer domains to the case of commutative rings with zero divisors. In this context there are investigated the following classes of commutative rings:

(I) Semihereditary rings.

(II) Rings which have weak dimension less or equal to one.

(III) Distributive rings.

(IV) Gaussian rings.

(V) Prüfer rings.

Recently the main stress in the area is focused on Gaussian rings (e.g. see [8] or [22]). In [22] S. Glaz showed that we have (I) $\Rightarrow$ (II) $\Rightarrow$ (III) $\Rightarrow$ (IV) $\Rightarrow$ (V) and no one of these implications can be replaced by the equivalence.

In this thesis the notion of a Gaussian ring is extended to the noncommutative setting by introducing a new class of rings which are called *right Gaussian rings*. We investigate the relations with noncommutative analogs of classes (I), (II), (III), (IV), and in some cases (V). Moreover, we study some related subjects which naturally occur during our research concerning right Gaussian rings.

In Chapter 2 we recall some facts regarding right distributive rings, and define right Gaussian rings. Moreover, we study basic properties of right Gaussian rings. We also present results about the connection between the above classes of rings.

Chapter 3 includes an investigation about right Gaussian skew power series rings. We will give an extension to the noncommutative case of a well-known result by Anderson and Camillo (see [2, Theorem 17]).

In Chapter 4 we define skew generalized power series rings and for positively ordered monoids we describe those of above which are right Gaussian.

It occurs that for a right Gaussian ring a ring of quotients may not exist, and even when it exists, it need not be right Gaussian. We study relevant these issues formulate Chapter 5.

In Chapter 6 we consider a class of homomorphic images of a polynomial ring $R[x]$ and give the necessary and sufficient conditions for a ring R under which these images are right Gaussian.

In Chapter 7 we make an effort to establish what kind of relations hold among right Gaussian rings, right Prüfer rings and some other classes of noncommutative rings.

Right Gaussian rings are exactly right duo Armendariz rings. This fact is a reason to take on Armendariz rings in detail, which we do in Chapter 8.

The final chapter contains investigations about some subclasses of unique product monoids which appear naturally in Chapter 8.

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Contents

Chapter 1

Preliminaries

1.1 Notation

In this section we want to introduce terminology and notations we will use throughout investigations.

1. Throughout, all rings are associative with unity, and if we say a *monoid* we will mean a semigroup with unity.

2. The sets of rational numbers, integers, and positive integers are denoted by $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$, respectively.

3. Throughout, for any subsets A, B of a ring R, and an element $r \in R$, $AB, A+B, rB$ and Br mean the following sets:

$AB = \{\sum_{i=1}^{n} a_i b_i : a_i \in A,\ b_i \in B \text{ for all } i, \text{ and } n \in \mathbb{N}\}$,

$A + B = \{a + b : a \in A, b \in B\}$,

$rB = \{\sum_{i=1}^{n} rb_i : b_i \in B \text{ for all } i, \text{ and } n \in \mathbb{N}\}$,

$Br = \{\sum_{i=1}^{n} b_i r : b_i \in B \text{ for all } i, \text{ and } n \in \mathbb{N}\}$,

4. (a) Let $(G, \cdot)$ and $(H, \star)$ be groups. A function $\varphi : G \to H$ is called a *group homomorphism* (or, *homomorphism of groups*) if $\varphi(a \cdot b) = \varphi(a) \star \varphi(b)$ for all $a, b \in G$.

(b) If R and P are rings, then a function $\varphi : R \to P$ is called a *ring homomorphism* (or, *homomorphism of rings*) if φ is a homomorphism of additive groups R and

P as well as $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(1) = 1$ for all $a, b \in R$. A homomorphism $\varphi : R \to R$ will be called an *endomorphism* of a ring R.

(c) Let R be a ring. By an *R-module homomorphism* we will mean a function $\varphi : M \to N$, where M and N are right R-modules, such that φ is a homomorphism of abelian groups M and N and $\varphi(m)r = \varphi(mr)$ for all $m \in M, r \in R$. Analogously we define R-module homomorphisms for left R-modules.

(d) Let $(S, \cdot)$ and $(T, \star)$ be monoids. A function $\varphi : S \to T$ is called a *monoid homomorphism* if $\varphi(a \cdot b) = \varphi(a) \star \varphi(b)$, and $\varphi(1) = 1$ for all $a, b \in S$.

5. (a) Let G, H be groups, and e_G, e_H identity elements of G and H, respectively. Then $\varphi : G \to H$, a homomorphism of groups G and H, is called *injective* if $\{a \in G : \varphi(a) = e_H\} = \{e_G\}$.

(b) Let R, P be rings. Then $\varphi : R \to P$, a ring homomorphism, is called *injective* if $\{a \in R : \varphi(a) = 0\} = \{0\}$. An injective homomorphism will be also called a *monomorphism* of rings. We will say that φ is *surjective* if $\{\varphi(a) : a \in R\} = P$. When it is said that φ is an *epimorphism*, then it means that φ is surjective homomorphism of rings. If φ is injective and surjective, then it is called *bijective*. Moreover if φ is bijective, then we will say that rings R and P are *isomorphic* and denote this fact by $R \cong P$.

(c) Let R be a ring, M and N be right (or, left) R-modules, and e_M, e_N identity elements of abelian groups M and N, respectively. Then an R-module homomorphism $\varphi : M \to N$ is called *injective* if $\{m \in M : \varphi(m) = e_N\} = \{e_M\}$. If $\{\varphi(m) : m \in M\} = N$, then we will say that φ is a *surjective R-homomorphism.*

6. For $\varphi : R \to P$, a ring homomorphism, the set $ker\varphi = \{r \in R : \varphi(r) = 0\}$ is called the *kernel of homomorphism* of φ, and the set $\varphi(R) = \{\varphi(r) : r \in R\}$ is called the *image of* φ.

7. Let R be a ring. A subset I of R is a *right ideal* of R if $a + b \in I$ and $ar \in I$ for all $a, b \in I$ and $r \in R$. A left ideal is defined similarly, and I is an ideal of R if it is a left and right ideal of R.

8. Let R be a ring, and $\{x_1, x_2, \ldots, x_n\} \subseteq R$ for some $n \in \mathbb{N}$. Then $(x_1, x_2, \ldots, x_n)_r$, $(x_1, x_2, \ldots, x_n)_l$, $(x_1, x_2, \ldots, x_n)$ denote respectively the right ideal, the left ideal and the ideal of R generated by the set $\{x_1, x_2, \ldots, x_n\}$.

9. If R is a ring and I is an ideal of R, then by R/I we will denote the factor ring of R by I. For an element $r \in R$, $\overline{r}$ will denote the image of r in R/I.

10. A ring R is called *domain* if for every nonzero $a, b \in R$, $ab \neq 0$.

11. A family $\{S_i : i \in I\}$ of subsets of a set S is said to satisfy the *Descending Chain Condition* (DCC) (resp. *Ascending Chain Condition* (ACC)) if there does not exist an infinite strictly descending chain $S_{i_1} \supsetneqq S_{i_2} \supsetneqq \ldots$ (resp. ascending chain $S_{i_1} \subsetneqq S_{i_2} \subsetneqq \ldots$) for some $S_{i_1}, S_{i_2}, \ldots$ where $i_1, i_2, \ldots \in I$.

12. A ring R is called *right Artinian (resp. left Artinian)* if the family of all right (resp. left) ideals of R satisfies DCC. A right and left Artinian ring will be called *Artinian.*

13. If a nonzero ring R has no ideal besides the zero ideal and itself, then we will say that R is *simple.* A ring R will be called *semisimple* if for every right (equivalently, left) ideal I of R, there exists a right (resp. left) ideal J such that $I \cap J = \{0\}$ and $R = I + J$.

14. Let R be a ring. If for every nonzero $a, b \in R$, $aRb \neq \{0\}$ (resp. $aRa \neq \{0\}$), then R is called *prime ring (resp. semiprime ring).* An ideal I of R is *prime (resp. semiprime)* if $I \neq R$ and R/I is prime (resp. semiprime) ring.

15. Let R be a ring. A right ideal I of R such that $I \neq R$ and I is not contained in any other right ideal different from R and itself, is called a *maximal right ideal.* A maximal left ideal is defined analogously. We will use freely the fact that every maximal ideal is a prime ideal.

16. For a ring R, $J(R)$ denotes the *Jacobson radical* of R, i.e. the intersection of all maximal right ideals of R (equivalently, the intersection of all maximal left ideals of R). If $J(R) = \{0\}$, then R is called *Jacobson semisimple.*

17. The *direct product* (resp. the *direct sum*) of rings R_i, for I some index set, will be denoted by $\prod_{i\in I} R_i$ (resp. $\bigoplus_{i\in I} R_i$). If $\prod_{i\in I} R_i$ is a direct product of rings, then for every $i_0 \in I$, $\pi_{i_0} : \prod_{i\in I} R_i \to R_{i_0}$ denotes the *natural projection.*, and $\tau_{i_0} : R_{i_0} \to \prod_{i\in I} R_i$ the *natural injection.*

18. Let R and $\{R_i : i \in I\}$ be rings. If for every $i \in I$ there exists a surjective ring homomorphism $\varphi_i : R \to R_i$, and $\bigcap_{i\in I} ker\varphi_i = \{0\}$, then we say that R can be represented as a *subdirect product* of $\{R_i : i \in I\}$ (or, R is a subdirect product of $\{R_i : i \in I\}$).

19. Let R be a ring. For a right R-module M and a left R-module N, by $M \otimes_R N$ we will denote the *tensor product* of modules M and N.

20. If α is a class of rings such that for every homomorphism of rings $\varphi : R \to S$, the fact that $R \in \alpha$ implies $\varphi(R) \in \alpha$, then we will say that the class α is *homomorphically closed.*

21. For a ring R and a subset X of R by $r_R(X)$ (resp. $l_R(X)$) we denote the right (resp. left) annihilator of X in R, i.e. $r_R(X) = \{r \in R : Xr = \{0\}\}$ (resp. $l_R(X) = \{r \in R : rX = \{0\}\}$).

22. For a ring or a monoid R, we will consider the following sets $U(R) = \{r \in R : rs = sr = 1, \text{ for some } s \in R\}$, and $Z(R) = \{r \in R : rs = sr, \text{ for every } s \in R\}$. Elements of $U(R)$ (resp. $Z(R)$) are called *units* (resp. *central elements*) of R.

1.2 Some constructions of rings

1. (a) For a ring R and a given endomorphism σ of R, $R[[x;\sigma]]$ (resp. $R[x;\sigma]$) is a ring whose elements are power series (resp. polynomials) in x, with coefficients in R written on the left, and with multiplication defined by $xa = \sigma(a)x$ for any $a \in R$. The above ring is called the *skew power series ring* (resp. *skew polynomial ring.*)

(b) If R is a ring and σ is an automorphism of R, then $R((x;\sigma))$ (resp. $R(x;\sigma)$) denotes the *skew Laurent series ring* (resp. *skew Laurent polynomial ring*) with coefficient ring R. This ring is formed by all series $f = \sum_{i=k}^{+\infty} a_i x^i$ (resp. polynomials $f = \sum_{i=k}^{n} a_i x^i$), where x is a variable, k is an integer (possibly negative), n is a positive integer, and all the coefficients a_i are contained in the ring R. In the ring $R((x;\sigma))$ (resp. $R(x;\sigma)$), addition is naturally defined and multiplication is defined by the formula $xa = \sigma(a)x$ (for all elements $a \in A$). For $\sigma = id_R$, we obtain the ordinary Laurent series ring $R((x))$ (resp. Laurent polynomial ring $R(x)$).

2. If $(R, +, \cdot)$ is a ring, then we can consider the set $R^{op} = R$ with two operations $+_{op}$ and $\cdot_{op}$ such that for every $a, b \in R^{op}$, $a +_{op} b = a + b$ and $a \cdot_{op} b = ba$. It is easy to check that then we get a ring which is called the *opposite ring* of R and is denoted by R^{op}.

3. Let R be a ring, M an (R, R)-bimodule, and $A = R \propto M$, the set of pairs (r, m) with $r \in R$ and $m \in M$, under coordinatewise addition and under an adjusted multiplication defined by $(r, m)(r', m') = (rr', rm' + mr')$ for all $r, r' \in R$, $m, m' \in M$. Then A has a structure of a ring and is called the *trivial ring extension of R by M*.

4. (a) Let R be a ring and S a multiplicative set in R (i.e. $S \cdot S \subseteq S$, $1 \in S$, and $0 \notin S$). Then a ring R_S is called a *right ring of quotients of R with respect to S* if there exists a ring homomorphism $\varphi : R \to R_S$ such that

(a) For any $s \in S$, $\varphi(s)$ is a unit of R_S.

(b) Every element of R_S has the form $\varphi(a)\varphi(s)^{-1}$ for some $a \in R$ and $s \in S$.

(c) $\ker \varphi = \{r \in R : rs = 0 \text{ for some } s \in S\}$.

It is well known (e.g. see [41, Theorem 10.6]) that the ring R has a right ring of quotients with respect to S if and only if the following conditions are satisfied:

(1) For any $a \in R$ and $s \in S$, $aS \cap sR \neq \emptyset$.

(2) For $a \in R$, if $ta = 0$ for some $t \in S$, then $as = 0$ for some $s \in S$.

A multiplicative set S satisfying the above conditions (1) and (2) is called a *right denominator set.* If the set S consists of all regular elements of R (i.e. all elements $a \in R$ such that a is neither a left zero-divisor nor a right zero-divisor of R), then the right ring of quotients R_S is called the *classical right ring of quotients of* R and is denoted by $Q^r_{cl}(R)$. Accordingly we can define *left denominator set* and *classical left ring of quotients of* R, $Q^l_{cl}(R)$. It is well known that if $Q^r_{cl}(R)$ and $Q^l_{cl}(R)$ exist, then $Q^r_{cl}(R) \cong Q^l_{cl}(R)$. In such case we will put $Q_{cl}(R)$.

(b) If R is a commutative ring, and P is a prime ideal of R, then it is easy to see that $S = R \backslash P$ is a multiplicative set in R. In such case we will call the ring R_S the *localization of* R *with respect to the ideal* P.

(c) For a commutative domain R, the localization R_S of R with respect to the ideal $\{0\}$ will be called the *field of fractions of* R. It is obvious that then $R_S = Q_{cl}(R)$.

1.3 Goldie rings

A nonzero right ideal I of a ring R is called *right uniform* if $aR \cap bR \neq \{0\}$ for any $a, b \in I \setminus \{0\}$. If for every right ideal J of R we have $I \cap J \neq \{0\}$, then I is called *right essential.*

Definition 1.3.1. A ring R has finite *right Goldie dimension* equal to n, for some $n \in \mathbb{N}$, if there exist right uniform ideals $I_1, \ldots, I_n$ of R such that $I_1 \oplus \cdots \oplus I_n$ is a direct sum (direct sum of right R-modules $I_1, ..., I_n$) and $I_1 + \cdots + I_n$ is an essential right ideal of R.

A ring R is said to satisfy *the ascending chain condition on right annihilators* if the family of all annihilators $\{r_R(\{a\}) : a \in R\}$ satisifies the ascending chain condition.

Definition 1.3.2. A ring R which satisfies the ascending chain condition on right annihilators and has finite right Goldie dimension is called a *right Goldie ring.* A left Goldie ring is defined adequately, and if a ring R is right and left Goldie, then it is simply called a *Goldie ring.*

Now we are in a position to present *Goldie's theorem* (see [46, Theorem 2.3.6]).

Theorem 1.3.3. *(Goldie's theorem) The following conditions are equivalent for a ring R.*

(1) R is a semiprime right Goldie ring.

(2) R is semiprime, there does not exists nonzero $a \in R$ such that $r_R(\{a\})$ is essential right ideal of R, and R has finite right Goldie dimension.

(3) R has the classical right ring of quotients $Q^r_{cl}(R)$, which is semisimple Artinian.

Moreover, the ring R is prime if and only if R has the classical right ring of quotients which is simple Artinian.

Chapter 2

Basic properties of right Gaussian rings

This Chapter is based on part of:

• R. Mazurek, M. Ziembowski, *Right Gaussian rings and skew power series rings*, submitted.

In this chapter we will define right distributive, and right Gaussian rings. Since right Gaussian rings are a new class of rings and are the main object of our interest, we will focus on this subject. We will also present some conditions which the above classes satisfy and give results regarding connection between them.

2.1 Right distributive rings

The first class of rings we want to talk about is the class of right distributive rings which were introduced in [72] by W. Stephenson.

Definition 2.1.1. A ring R is *right distributive* if for any right ideals I, J, K of R, $(I + J) \cap K = (I \cap K) + (J \cap K)$.

The left distributive rings can be definied analogously and rings which are left and right distributive are called distributive.

In our investigations we will often use the following characterization of right distributive rings (see [72, Theorem 1.6]).

Theorem 2.1.2. *A ring R is right distributive if and only if for any $a, b \in R$ there exists $f \in R$ such that $af \in bR$ and $b(1-f) \in aR$.*

Since the moment when Stephenson's paper was published, many authors have been interested in the right distributive rings. For example these rings were investigated in [18] where the authors proved the following (see [18, Corollary 3.6]):

Theorem 2.1.3. *A ring R is right distributive if and only if for any $a, b \in R$ and any maximal right ideal M of R there exists $s \in R \setminus M$ such that $as \in bR$ or $bs \in aR$.*

2.2 Right Gaussian rings

Now it is time to introduce the class of right Gaussian rings which are the main object of our interest in the thesis.

For a ring R and a polynomial $f \in R[x]$, let $c_r(f)$ denote the right ideal of R generated by the coefficients of f. Obviously, for any $f, g \in R[x]$ we have $c_r(fg) \subseteq c_r(f)c_r(g)$.

Definition 2.2.1. A ring R is *right Gaussian* if $c_r(f)c_r(g) = c_r(fg)$ for any $f, g \in R[x]$.

We start the study of right Gaussian rings by observing that the class of these rings is homomorphically closed.

Proposition 2.2.2. *If a ring R is right Gaussian, then so is any homomorphic image of R.*

Proof. Let $\varphi : R \to R'$ be a ring epimorphism. For any $f = \sum_{i=0}^{m} a_i x^i \in R[x]$ we set $\overline{\varphi}(f) = \sum_{i=0}^{m} \varphi(a_i)x^i \in R'[x]$, obtaining a ring epimorphism $\overline{\varphi} : R[x] \to R'[x]$. Since $c_r(\overline{\varphi}(f)) = \varphi(c_r(f))$ for any $f \in R[x]$, the result follows. □

Corollary 2.2.3. *A direct product ring $\prod_{i\in I} R_i$ is right Gaussian if and only if each component ring R_i is.*

Proof. The "only if" part follows from Proposition 2.2.2 applied to the natural projection of $\prod_{i\in I} R_i$ on R_i. To prove the converse, consider any polynomial $h = \sum_{j=0}^{m} a_j x^j \in (\prod_{i\in I} R_i)[x]$, and for any $i_0 \in I$ set $h_{i_0} = \sum_{j=0}^{m} \pi_{i_0}(a_j)x^j \in R_{i_0}[x]$, where $\pi_{i_0} : \prod_{i\in I} R_i \to R_{i_0}$ is the natural projection. It is easy to see that $c_r(h) = \prod_{i\in I} c_r(h_i)$. Hence, if all the R_i's are right Gaussian, then for any $f, g \in (\prod_{i\in I} R_i)[x]$ we have

$$c_r(f)c_r(g) = \prod_{i\in I} c_r(f_i) \cdot \prod_{i\in I} c_r(g_i) \subseteq \prod_{i\in I} c_r(f_i)c_r(g_i) =$$

$$= \prod_{i\in I} c_r(f_i g_i) = \prod_{i\in I} c_r((fg)_i) = c_r(fg).$$

Since the inclusion $c_r(fg) \subseteq c_r(f)c_r(g)$ is obvious, we obtain $c_r(f)c_r(g) = c_r(fg)$, which proves the "if" part. □

Remark 2.2.4. At the end of the next chapter we will be able to construct a ring (see Example 3.7.5) which is a subdirect product of right Gaussian rings but is not right Gaussian itself. This implies that regarding a subdirect product and right Gaussianess, we have different situation than for a direct product.

Recall that a ring R is *right duo* (resp. *right quasi-duo*) if every right ideal of R (resp. every maximal right ideal of R) is two-sided. It is easy to see that a ring R is right duo if and only if every principal right ideal of R is ideal. R is duo if it is right and left duo.

Now, we have the lemma which implies many consequences for our investigations.

Lemma 2.2.5. *If a ring R is right Gaussian, then R is right duo.*

Proof. Let I be a right ideal of R, let $a \in I$, and let $f = 1, g = a \in R[x]$. Then $c_r(f) = R$ and $c_r(g) = c_r(fg) = aR$, and since R is right Gaussian, we obtain $RaR = c_r(f)c_r(g) = c_r(fg) = aR$. Hence $Ra \subseteq aR \subseteq I$, proving that R is right duo. □

Recall that a ring R is an *Armendariz ring* if whenever the product of two polynomials over R is zero, then the products of their coefficients are all zero, that is, for any $f = \sum_{i=0}^{m} a_i x^i,\ g = \sum_{j=0}^{n} b_j x^j \in R[x]$, if $fg = 0$, then $a_i b_j = 0$ for all i, j.

It is obvious that any right Gaussian ring is Armendariz. As proved by Anderson and Camillo in [2, Theorem 8], a commutative ring R is Gaussian if and only if every homomorphic image of R is Armendariz. Below we extend the Anderson-Camillo result to noncommutative rings.

Theorem 2.2.6. *A ring R is right Gaussian if and only if R is right duo and every homomorphic image of R is Armendariz.*

Proof. The "only if" part follows from Lemma 2.2.5 and Proposition 2.2.2. To prove the "if" part, consider any polynomials $f = \sum_{i=0}^{m} a_i x^i,\ g = \sum_{j=0}^{n} b_j x^j \in R[x]$. Since R is right duo, $I = c_r(fg)$ is an ideal of R. In what follows the "bars" refer to modulo I, that is $\overline{R} = R/I$, and $\overline{a} = a + I$ for any $a \in R$. Clearly, all the coefficients of the product fg belong to $c_r(fg)$, and thus for the polynomials $\hat{f} = \sum_{i=0}^{m} \overline{a}_i x^i,\ \hat{g} = \sum_{j=0}^{n} \overline{b}_j x^j \in \overline{R}[x]$ we have $\hat{f}\hat{g} = 0$. Since $\overline{R}$ is Armendariz, it follows that $\overline{a}_i \overline{b}_j = 0$ for any i, j, and thus $a_i b_j \in I = c_r(fg)$. Since R is right duo, we get $a_i R b_j \subseteq a_i b_j R \subseteq c_r(fg)$, which leads to $c_r(f) c_r(g) = c_r(fg)$. □

At this moment we would like to stress that in the Chapter 8 we will consider Armendariz rings and their generalizations more attentively.

In the remaining part of the section we will present results that provide us with examples of right Gaussian rings and show relations between right distributive rings and right Gaussian rings.

In Proposition 2.2.7 below we characterize the right Gaussianess of the skew polynomial ring $R[x; \sigma]$.

Recall that a ring R is said to be *von Neumann regular* if $a \in aRa$ for any $a \in R$.

Proposition 2.2.7. *Let σ be an endomorphism of a ring R. Then the following conditions are equivalent:*

(1) $R[x;\sigma]$ is right Gaussian.

(2) $R[x;\sigma]$ is right distributive and σ is injective.

(3) R is commutative von Neumann regular and σ is the identity on R.

Proof. (1) $\Rightarrow$ (3) Assume that $R[x;\sigma]$ is right Gaussian. Then by Lemma 2.2.5, $R[x;\sigma]$ is right duo, and thus [51, Theorem 1] implies that R is commutative and σ is the identity map of R. Hence $R[x]$ is a commutative Gaussian ring, and applying [2, Theorem 16] we obtain that R is von Neumann regular.

(2) $\Leftrightarrow$ (3) This equivalence is proved in [80, 6.67].

(3) $\Rightarrow$ (1) Follows from [2, Theorem 16]. □

The following example shows that for a right Gaussian ring R, neither the polynomial ring $R[x]$ nor subrings of R must be right Gaussian, even in the case where R is commutative.

Example 2.2.8. (a) Let F be a field, and let $R = F[y]$. Then by Proposition 2.2.7, R is Gaussian but by the same proposition, the polynomial ring $R[x]$ is not Gaussian.

(b) Let $R = F(x,y)$ be the rational function field in two variables x, y over a field F. Then R is Gaussian, but by Proposition 2.2.7 the subring $S = F[x,y]$ of R is not Gaussian. □

In an obvious way one can define left Gaussian rings. It will be shown in Example 2.2.10 that a right Gaussian ring need not be left Gaussian. To construct the example, we will use the following property of local rings. Recall that a ring R is *local* if R has exactly one maximal right (left) ideal; in this case the unique maximal right (left) ideal of R coincides with the Jacobson radical $J(R)$ of R.

Proposition 2.2.9. *Let R be a local ring with $J(R)^2 = \{0\}$. Then*

(i) R is Armendariz.

(ii) If R is right duo, then R is right Gaussian.

Proof. (i) Set $J = J(R)$ and consider any polynomials $f = \sum_{i=0}^{m} a_i x^i$, $g = \sum_{j=0}^{n} b_j x^j \in R[x]$ with $fg = 0$. Since the ring R/J is a domain, so is the ring $R[x]/J[x] \cong (R/J)[x]$, and thus $f \in J[x]$ or $g \in J[x]$. If $f \in J[x]$ and $g \in J[x]$, then using $J^2 = \{0\}$, we obtain $a_i b_j = 0$ for any i, j. Next we consider the case where $f \in J[x]$ and $g \notin J[x]$. If $f \neq 0$, we can choose minimal i_0 with $a_{i_0} \neq 0$, and minimal j_0 with $b_{j_0} \notin J$. Then $a_{i_0} b_{j_0}$ is the $x^{i_0+j_0}$-coefficient of fg, and thus $a_{i_0} b_{j_0} = 0$. Since by [39, Theorem 19.1] b_{j_0} is a unit of R, we obtain $a_{i_0} = 0$, a contradiction. Hence $f = 0$, and thus $a_i b_j = 0$ for all i, j. Similarly one concludes that all the $a_i b_j$'s are zero in the remaining case where $f \notin J[x]$ and $g \in J[x]$.

(ii) Since the class of local right duo rings R with $J(R)^2 = \{0\}$ is homomorphically closed, (ii) follows from (i) and Theorem 2.2.6. □

Now we are in a position to construct a right Gaussian ring that is not left Gaussian.

Example 2.2.10. Let F be a field and φ an endomorphism of F such that $\varphi(F) \neq F$. Let R be the set of all matrices of the form $\left(\begin{smallmatrix} \varphi(a) & b \\ 0 & a \end{smallmatrix}\right)$, where $a, b \in F$. Then R is a ring with usual addition and multiplication of matrices as operations. Since the only right ideals of R are 0, $J(R) = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$ and R itself, it follows that R is a local right duo ring. Since furthermore $J(R)^2 = \{0\}$, R is right Gaussian by Proposition 2.2.9. On the other hand, since $F \nsubseteq \varphi(F)$, also $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) R \nsubseteq R \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and thus R is not left duo. Hence the left version of Lemma 2.2.5 implies that R is not left Gaussian. □

In Theorem 2.2.11 below, we present what kind of relations hold between right distributive rings and right Gaussian rings. To get the main result of the section, however, it is not possible to use the rings of quotients, which are the main tool in studying commutative Gaussian rings, since in general the rings of quotients do not exist for noncommutative right Gaussian rings, as will be shown in Chapter 5.

Theorem 2.2.11. *If R is a right duo right distributive ring, then R is right Gaussian.*

Proof. Let R be a right duo right distributive ring. Since the class of such rings is homomorphically closed, it follows from Theorem 2.2.6 that to prove that R is right Gaussian, it suffices to show that R is an Armendariz ring.

Let $f = \sum_{k=0}^{m} a_k x^k, g = \sum_{l=0}^{n} b_l x^l \in R[x]$ be any polynomials such that $fg = 0$. We have to show that $a_k b_l = 0$ for any k, l. For this, it suffices to prove that

$$\text{for any maximal right ideal } M \text{ of } R \text{ there exists } s \in R \setminus M \text{ with } a_k b_l s = 0. \tag{2.2.1}$$

Indeed, assume (2.2.1) and suppose that $a_k b_l \neq 0$ for some k, l. Then $I = \{x \in R : a_k b_l x = 0\}$ is a right ideal of R which is proper (since $1 \notin I$), and thus $I \subseteq M$ for some maximal right ideal M of R, which would contradict (2.2.1).

To prove (2.2.1), consider any maximal right ideal M of R. Since the ring R is right distributive, it follows from [72, Corollary 4 of Proposition 1.1] and [18, Corollary 3.6] that M is an ideal of R, the set $R \setminus M$ is multiplicatively closed, and

$$\text{for any } c, d \in R \text{ there exists } t \in R \setminus M \text{ such that } ct \in dR \text{ or } dt \in cR.$$

Using that, along with the assumption that R is a right duo ring, it easily follows by induction on n that there exists $j \in \{0, 1, \ldots, n\}$ such that for some $u \in R \setminus M$ we have $b_l u \in b_j R$ for any $l \in \{0, 1, \ldots, n\}$. Choose maximal j with the above property, and consider any $p \in \{0, 1, \ldots, n\}$ with $p > j$. If $b_p u \notin b_j M$, then since $b_p u \in b_j R$, $b_p u = b_j w$ for some $w \in R \setminus M$. Hence $uw \in R \setminus M$ and for any $l \in \{0, 1, \ldots, n\}$ we have $b_l uw \in b_j Rw \subseteq b_j wR = b_p uR \subseteq b_p R$, a contradiction with the maximality of j. Thus $b_p u \in b_j M$. Hence, as we have just proved, there exists $j \in \{0, 1, \ldots, n\}$ and $u \in R \setminus M$ such that for any $l \in \{0, 1, \ldots, n\}$,

$$b_l u \in b_j R, \text{ and furthermore, if } l > j, \text{ then } b_l u \in b_j M. \tag{2.2.2}$$

In what follows, j and u is a concrete pair satisfying (2.2.2).

Next we consider the coefficients of the polynomial $f = \sum_{k=0}^{m} a_k x^k \in R[x]$. Since R is right duo, for any k there exists $a'_k \in R$ with $a_k b_j = b_j a'_k$. Similarly as above, one can show that there exists $i \in \{0, 1, \ldots, m\}$ and $v \in R \setminus M$ such that for any $k \in \{0, 1, \ldots, m\}$,

$$a'_k v \in a'_i R, \text{ and furthermore, if } k > i, \text{ then } a'_k v \in a'_i M. \tag{2.2.3}$$

We are now ready to complete the proof. Since $fg = 0$, the x^{i+j}-coefficient of fg is equal to 0, and thus the coefficient multiplied by uv is equal to 0 as well, i.e.

$$\sum_{\{(k,l):\ k+l=i+j\}} a_k b_l uv = 0. \tag{2.2.4}$$

Let $a_k b_l uv$ be any summand of (2.2.4) with $(k, l) \neq (i, j)$. Then either $k > i$ or $l > j$. If $k > i$, then using (2.2.2), (2.2.3) and the right duo condition, we obtain

$$a_k b_l uv \in a_k b_j Rv = b_j a'_k Rv \subseteq b_j a'_k vR \subseteq b_j a'_i MR \subseteq a_i b_j M.$$

Since R is right duo and $v \in R \setminus M$, it follows that $Mv \subseteq vM$, and thus in the case where $l > j$, we obtain

$$a_k b_l uv \in a_k b_j Mv = b_j a'_k Mv \subseteq b_j a'_k vM \subseteq b_j a'_i RM \subseteq a_i b_j M.$$

Therefore, for any summand $a_k b_l uv$ of (2.2.4) with $(k, l) \neq (i, j)$ we have $a_k b_l uv \in a_i b_j M$. Hence, it follows from (2.2.4) that for some $q \in M$ we have $a_i b_j uv + a_i b_j q = 0$, and thus $a_i b_j t = 0$ with $t = uv + q \in R \setminus M$. Set $s = uvt$, and consider any pair a_k, b_l of the coefficients of the polynomials f and g. Then $s \in R \setminus M$, and

$$a_k b_l s = a_k b_l uvt \in a_k b_j Rvt = b_j a'_k Rvt \subseteq b_j a'_k vRt \subseteq b_j a'_i Rt = a_i b_j Rt \subseteq a_i b_j tR = \{0\},$$

which proves (2.2.1). □

A ring R is a *right chain ring* (resp. *left chain ring*) if its right (resp. left) ideals are totally ordered by inclusion ([9]). Right and left chain ring is called *chain*. Clearly, any right chain ring is right distributive.

Remark 2.2.12. The ring A from Example 7.1.6 is not right duo but is a chain domain. So A is right and left distributive but not right Gaussian by Lemma 2.2.5. Therefore the assumption in Theorem 2.2.11 that R is right duo is not superfluous.

In the following example we construct a commutative Gaussian ring that is not distributive (cf. [22, Example 3.3.2]). Hence the converse of Theorem 2.2.11 is not true, even in the case of commutative rings.

Example 2.2.13. Let V be a vector space over a field F such that $\dim_F V \geqslant 2$. Then the set R of all matrices of the form $\left(\begin{smallmatrix} a & v \\ 0 & a \end{smallmatrix}\right)$, where $a \in F$ and $v \in V$, with usual addition and multiplication of matrices, is a commutative local ring with $J(R) = \left(\begin{smallmatrix} 0 & V \\ 0 & 0 \end{smallmatrix}\right)$. Since $J(R)^2 = \{0\}$, R is Gaussian by Proposition 2.2.9(ii). On the other hand, since $\dim_F V \geqslant 2$, we can choose two linearly independent vectors $u, v \in V$. Since for the ideals $I = \left(\begin{smallmatrix} 0 & Fu \\ 0 & 0 \end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix} 0 & Fv \\ 0 & 0 \end{smallmatrix}\right)$ and $K = \left(\begin{smallmatrix} 0 & F(u+v) \\ 0 & 0 \end{smallmatrix}\right)$ of R we have

$$(I + J) \cap K = K \text{ and } (I \cap K) + (J \cap K) = 0,$$

R is not distributive. $\square$

Recall that an element e of a ring R is called *idempotent* if $e = e^2$. If e, f are idempotents of a ring R such that $ef = fe = 0$, then we say that e and f are *orthogonal idempotents* in R. A ring R is *strongly regular* if $a \in a^2R$ for any $a \in R$. We have the following:

Lemma 2.2.14. *A ring R is strongly regular if and only if for every $a \in R$ there exist $u \in U(R)$ and idempotent $e \in Z(R)$ such that $a = ue$.*

Proof. If R is strongly regular, then by [40, Ex. 12.6C] for every $a \in R$ there exist $u \in U(R)$ such that $a = aua$. Then $uaua = ua$ is an idempotent in R. By [40, Ex. 12.6A] ua is central. Hence $u^{-1}ua$ is required form of a.

If for $a \in R$ there exist $u \in U(R)$ and idempotent $e \in Z(R)$ such that $a = ue$, then $a = ueuu^{-1}e = ueueu^{-1} = a^2u^{-1}$, so the fact that R is strongly regular follows. $\square$

Now it is easy to see that if a ring R is strongly regular, then $J(R) = \{0\}$, and that strongly regular rings are precisely right duo von Neumann regular rings.

Since strongly regular rings are right distributive, the following corollary is an immediate consequence of Theorem 2.2.11.

Corollary 2.2.15. *Strongly regular rings are right Gaussian.*

The following corollary follows immediately from Theorem 2.2.11.

Corollary 2.2.16. *If R is a right duo right chain ring, then R is right Gaussian.*

Chapter 3

Right Gaussian skew power series rings

This Chapter is based on:

• R. Mazurek, M. Ziembowski, *Duo, Bézout and distributive rings of skew power series*, Publ. Mat. 53 (2009), no. 2, 257–271.

and on part of:

• R. Mazurek, M. Ziembowski, *Right Gaussian rings and skew power series rings*, submitted.

In this chapter we characterize skew power series rings that are right Gaussian, extending to the noncommutative case a well-known result by Anderson and Camillo (see [2, Theorem 17]).

3.1 Results on skew power series rings

In present section we would like to recall those results on (skew) power series rings that are an important part of our motivation and direction of studies in this thesis.

Recall that a ring R is *right self-injective* (resp. *left self-injective*) if any R-module homomorphism from a right (resp. left) ideal I of R into R (I and R are considered as right R-modules) extends to an R-module homomorphism from R

into R (R is seen as a right (resp. left) R-module). A right module M over a ring R is *countable-injective* if any R-module homomorphism from a countably generated right ideal of R into M extends to an R-module homomorphism of R into M. If a ring R is countable-injective as a right (resp. left) R-module, then we will say that R is right (resp. left) countable-injective ring. A ring R which is left and right countable-injective is called *countable-injective.* A ring R is said to be *right countable-algebraically compact* if for any system of a countable number of linear equations with a countable number of indeterminates and with coefficients from R written on the left, if every finite subsystem of this system has a solution in R, then the whole system has a solution in R. This is to say that if A is an $\aleph_0 \times \aleph_0$ row-finite matrix over R, indexed by $\mathbb{N} \times \mathbb{N}$, X is a column of $\aleph_0$ indeterminates and B is a column of $\aleph_0$ elements of R, both indexed by $\mathbb{N}$, then the system $AX = B$ is finitely solvable if and only if it is solvable. Left countable-algebraically compact rings are defined analogously. It is well known that for a von Neumann regular ring R, the ring R is right (left) countable-algebraically compact if and only if it is left (right) countable-injective (e.g. see [26, Proposition 1.1]). During our studies we will need the following well known result (e.g. see [80, 4.88]).

Proposition 3.1.1. *For a strongly regular ring R, the following conditions are equivalent.*

(1) R is right countable-injective.

(2) R is left countable-injective.

(3) R is right countable-algebraically compact.

(4) R is left countable-algebraically compact.

(5) Each factor ring $\overline{R}$ of R is a countable-injective countable-algebraically compact ring, and each cyclic right (or left) $\overline{R}$ - module is countable-injective.

(6) For each sequence $(e_n)_{n=0}^{\infty}$ of mutually orthogonal idempotents of R and for

each sequence $(a_n)_{n=0}^{\infty}$ *of elements of* R *there exists* $b \in R$ *such that* $be_n = a_n e_n$ *for all* $n \in \mathbb{N} \cup \{0\}$.

During our investigations we will consider Bézout rings which are defined as follows.

Definition 3.1.2. A ring R is *right Bézout* if every finitely generated right ideal of R (equivalently, every right ideal of R generated by two elements) is principal.

Now, we would like to present some well known connections between right distributive rings and right Bézout rings. We collect them in the following:

Proposition 3.1.3. *Let* R *be a ring. Then*

(i) ([80, 5.16(1)]) *If the factor ring* $R/J(R)$ *is strongly regular, then* R *is right distributive if and only if* R *is right Bézout.*

(ii) ([80, 2.35]) *If* R *is right Bézout and right quasi-duo, then* R *is right distributive.*

By the next example it follows that the classes of right distributive (resp. right Gaussian) rings and right Bézout rings are different to each other.

Example 3.1.4. (a) ([81]) The ring $R = \mathbb{Z} + \mathbb{Z}\sqrt{-5}$ is commutative distributive domain which is neither right nor left Bézout. By Theorem 2.2.11 the ring R is right and left Gaussian as well.

(b) ([81]) Let F be a field. Then for every $n > 1$, the ring $M_n(F)$ of all $n \times n$ matrices over F is Bézout but neither right nor left distributive.

(c) Let F be a field. By (b) for every $n > 1$, the ring $R = M_n(F)$ is Bézout. Let us consider two elements $a, b \in R$ such that the only nonzero entry of a is $(1, n)$, and the only nonzero entry of b is $(1, 1)$, and these entries are equal to 1. Then $a, b \in R[x]$ and $ab = 0$. So $c_r(ab) = 0$.

It is easy to see that $b \in aR$. Therefore, $b = b^2 \in aRbR = c_r(a)c_r(b)$. Since $b \neq 0$ and $c_r(ab) = 0$ it follows that the ring R is not right Gaussian. Analogously we can show that R is not left Gaussian. □

We will also need the following two definitions.

Definition 3.1.5. A right R-module M is called *flat*, if the fact that $\varphi : A \to B$ is an injective homomorphism of left R-modules, implies that the map

$$id_M \otimes \varphi : M \otimes_R A \to M \otimes_R B,$$

such that for any $m \in M, a \in A$, $(id_M \otimes \varphi)(m \otimes a) = m \otimes \varphi(a)$, is an injective homomorphism of abelian groups.

For our needs we have the following (see [41, Examples 5.62a, 5.62b]):

Definition 3.1.6. We will say that a ring R has *weak dimension less or equal to one* if either R is von Neumann regular, or R is not von Neumann regular and any right (equivalently, left) ideal of R is flat as a right (resp. left) R-module.

In [10] J. Brewer, E. Rutter and J. Watkins proved that for any commutative power series ring $R[[x]]$ the following equivalences hold:

$$\begin{gathered} R[[x]] \text{ is Bézout} \\ \Updownarrow \\ R \text{ is countable-injective von Neumann regular} \\ \Updownarrow \\ R[[x]] \text{ has weak dimension less or equal to one.} \end{gathered}$$

Bézout power series rings in the noncommutative setting were studied by D. Herbera in [26], where among other results she proved a generalization of the Brewer, Rutter and Watkins result, which we quote below.

Theorem 3.1.7. (Herbera; see [26, Corollary 1.10]) *Let R be a strongly regular ring. The following conditions are equivalent:*

(1) $R[[x]]$ is right Bézout.

(2) $R[[x]]$ is Bézout.

(3) $R[[x]]$ has weak dimension less or equal to one.

(4) R is countable-injective.

In this situation $R[[x]]$ is duo and all ideals of $R[[x]]$ are generated by central elements.

Duo Bézout power series rings appeared also in a characterization of countable-injective strongly regular rings given by O.A.S. Karamzadeh and A.A. Koochakpoor in [34], where they proved the following theorem.

Theorem 3.1.8. (Karamzadeh and Koochakpoor; see [34, Theorem 1.10]) *Let R be a strongly regular ring. Then R is countable-injective if and only if $R[[x]]$ is duo and Bézout.*

In [51] G. Marks, among other results, proved the following theorem.

Theorem 3.1.9. (Marks; see [51, Proposition 5]) *Let R be a left or right self-injective von Neumann regular ring. The following conditions are equivalent:*

(1) $R[[x]]$ is right duo.

(2) $R[[x]]$ is left duo.

(3) R is right duo.

(4) R is left duo.

In [77] and [78] A.A. Tuganbaev extended the previously mentioned result of Brewer, Rutter and Watkins to skew power series rings, adding additionally the right distributivity property to the list of equivalences, and showing that the condition concerning the weak dimension can be replaced with a weaker property of 2-generated right ideals (i.e., right ideals generated by two elements). To quote those results of Tuganbaev, we need a definition.

A ring R is called *semicommutative* if the right annihilator of every element of R is a two-sided ideal (clearly the notion is left-right symmetric).

Theorem 3.1.10. (Tuganbaev; see [78, Theorem 2]) *Let σ be an injective endomorphism of a ring R. Then the following conditions are equivalent:*

(1) $R[[x;\sigma]]$ is right Bézout and R is semicommutative.

(2) $R[[x;\sigma]]$ is right Bézout and R is right quasi-duo.

(3) $R[[x;\sigma]]$ is right distributive.

(4) R is countable-injective strongly regular, σ is bijective and $\sigma(e) = e$ for any idempotent $e = e^2 \in R$.

Recall that a ring R is *abelian* if all idempotents of R are central.

Theorem 3.1.11. (Tuganbaev; see [77, Theorem 1]) *Let R be an abelian ring and σ an automorphism of R such that $\sigma(e) = e$ for any idempotent $e = e^2 \in R$. Then the following conditions are equivalent:*

(1) $R[[x;\sigma]]$ has weak dimension less or equal to one.

(2) All 2-generated right ideals of $R[[x;\sigma]]$ are flat.

(3) R is countable-injective strongly regular.

In the present chapter we show that in some sense all the conditions appearing in the results quoted above (starting from that of Brewer, Rutter and Watkins) are equivalent. More precisely, these results are consequences of Theorem 3.6.1. It should be emphasized that besides conditions suggested by the quoted results, the theorem also contains some new ones. It should be also stressed that according to the second part of this theorem, each of the equivalent conditions implies its left analogue (but need not be equivalent to it; see Example 3.6.3), and if σ is an automorphism, then any condition in this theorem is equivalent to its left analogue. To prove this theorem we need some results on special classes of skew power series rings which we assemble in next few sections.

3.2 Right Bézout rings of skew power series

Later on we will need the following generalization of Herbera's result [26, Lemma 2.2]. Recall that a ring R is *Dedekind-finite* if for any $a, b \in R$, $ab = 1$ implies

$ba = 1$.

Proposition 3.2.1. *Let R be a Dedekind-finite ring and σ an injective endomorphism of R such that $R[[x;\sigma]]$ is right Bézout. Then R is von Neumann regular.*

Proof. Set $A = R[[x;\sigma]]$. For any $f = \sum_{n=0}^{\infty} f_n x^n \in A$ we set $\hat{f} = \sum_{n=0}^{\infty} \sigma(f_n)x^n$, i.e., $\hat{f}$ is a unique element of A such that $xf = \hat{f}x$. It is clear that $\widehat{fg} = \hat{f}\hat{g}$ for any $f, g \in A$.

We claim that (cf. [26, Proposition 2.1])

$$for\ any\ f = \sum_{n=0}^{\infty} f_n x^n,\ g = \sum_{n=0}^{\infty} g_n x^n \in A,\ if\ fg = x,\ then\ \hat{g}f = x.$$

To see this, note that since A is right Bézout, the right ideal $gA + xA$ is principal and thus there exist $h = \sum_{n=0}^{\infty} h_n x^n,\ k = \sum_{n=0}^{\infty} k_n x^n,\ \alpha = \sum_{n=0}^{\infty} \alpha_n x^n,\ \beta \in A$ such that

$$g = (gh + xk)\alpha \text{ and } x = (gh + xk)\beta. \tag{3.2.1}$$

Multiplying the first part of (3.2.1) on the left by f, we obtain $x = (xh + fxk)\alpha$, and equating x-coefficients, we see that

$$1 = [\sigma(h_0) + f_0\sigma(k_0)]\sigma(\alpha_0).$$

Since R is Dedekind-finite, it follows that $\sigma(\alpha_0)$ is a unit of R, and thus $\hat{\alpha}$ is a unit of A (see [39, p. 10]). Since (3.2.1) implies that

$$\hat{g} = (\widehat{gh + xk})\hat{\alpha} \text{ and } x = (\widehat{gh + xk})\hat{\beta}, \tag{3.2.2}$$

we deduce that

$$x = \hat{g}\gamma \text{ for } \gamma = \hat{\alpha}^{-1}\hat{\beta}. \tag{3.2.3}$$

By assumption $fg = x$, and thus $\hat{f}\hat{g} = x$. Hence (3.2.3) implies that $\hat{f}x = x\gamma = \hat{\gamma}x$, and $\hat{f} = \hat{\gamma}$ follows. Since σ is injective, $f = \gamma$, which proves our claim.

Now we are in a position to prove that R is von Neumann regular. Let $a \in R$.

Since A is right Bézout, the right ideal $aA + xA$ is principal, and thus for some $p = \sum_{n=0}^{\infty} p_n x^n$, q, $\delta = \sum_{n=0}^{\infty} \delta_n x^n$, $\epsilon = \sum_{n=0}^{\infty} \epsilon_n x^n \in A$ we have

$$a = (ap + xq)\delta \text{ and } x = (ap + xq)\epsilon. \tag{3.2.4}$$

Applying our claim to the second part of (3.2.4), we obtain $\hat{\epsilon}(ap + xq) = x$ and thus the first part of (3.2.4) implies the equality $\hat{\epsilon}a = x\delta$. Equating x-coefficients in the equality, we see that $\sigma(\epsilon_1)\sigma(a) = \sigma(\delta_0)$, and since σ is injective, it follows that $\delta_0 = \epsilon_1 a$. On the other hand, equating constant terms in the first part of (3.2.4), we obtain $a = ap_0\delta_0$. Hence $a = ap_0\delta_0 = ap_0\epsilon_1 a \in aRa$. □

3.3 Right duo rings of skew power series

In the proof of the following proposition we use an argument which is essentially due to Marks (see [51, proof of Theorem 1]).

An endomorphism σ of a ring R is *idempotent-stabilizing* if $\sigma(e) = e$ for every idempotent $e \in R$.

Proposition 3.3.1. *Let R be a ring and σ an endomorphism of R. If $R[[x;\sigma]]$ is right duo, then R is right duo, σ is bijective and idempotent-stabilizing.*

Proof. Set $A = R[[x;\sigma]]$. Since by assumption A is right duo, for any $a, b \in R$ there exists $f = \sum_{n=0}^{\infty} f_n x^n \in A$ such that $ba = af$. Hence $ba = af_0 \in aR$, proving that R is right duo.

Since A is right duo, for any $a \in R$ there exists $g \in A$ such that $ax = xg$. Equating x-coefficients in this equality, we obtain $a \in \sigma(R)$, and thus σ is surjective.

Now we show that σ is injective. Let $a \in R$ be such that $\sigma(a) = 0$. Since σ is surjective, $a = \sigma(b)$ for some $b \in R$. Since A is right duo, in R there exists a sequence $(c_n)_{n\in\mathbb{N}}$ such that

$$ax + x^3 = x(b + ax + x^2) = (b + ax + x^2)(c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots). \tag{3.3.1}$$

Equating constant terms, x-, and x^3-coefficients in (3.3.1), we obtain the following equations:

$$0 = bc_0, \quad a = bc_1 + a\sigma(c_0), \quad 1 = bc_3 + a\sigma(c_2) + \sigma^2(c_1). \tag{3.3.2}$$

From the first part of (3.3.2) we obtain $0 = \sigma(bc_0) = a\sigma(c_0)$, and thus the second part of (3.3.2) implies that $a = bc_1$, which leads to $0 = a\sigma(c_1)$. Applying σ^2 to the third equation of (3.3.2), we obtain $\sigma^4(c_1) = 1$. Hence, since A is right duo, it follows that $x^3 = x^3\sigma(c_1) \in \sigma(c_1)A$, and thus $1 = \sigma(c_1)d$ for some $d \in R$. Hence $a = a(\sigma(c_1)d) = (a\sigma(c_1))d = 0$, proving that σ is bijective.

Finally we show that σ if idempotent-stabilizing. Since A is right duo, there exist $f, g \in A$ such that $\sigma(e)x = xe = ef$ and $\sigma(1-e)x = x(1-e) = (1-e)g$. Thus there exist $a, b \in R$ such that

$$\sigma(e) = ea \quad \text{and} \quad 1 - \sigma(e) = \sigma(1-e) = (1-e)b. \tag{3.3.3}$$

From the second part of (3.3.3) it follows that $e = e\sigma(e)$. Hence the first part of (3.3.3) implies that $e = e\sigma(e) = \sigma(e)$. □

3.4 Skew power series rings over countable-injective rings

In the proof of Theorem 3.6.1 we will also need the following:

Proposition 3.4.1. *If R is a countable-injective strongly regular ring and σ is an idempotent-stabilizing automorphism of R, then any principal right ideal of the ring $R[[x;\sigma]]$ is generated by a power series whose coefficients are mutually orthogonal idempotents of R.*

Proof. Set $A = R[[x;\sigma]]$ and let $f = \sum_{n=0}^{\infty} f_n x^n \in A$. Since R is a strongly regular ring, by Lemma 2.2.14 for any $n \in \mathbb{N} \cup \{0\}$ there exist a central idempotent $d_n \in R$ and a unit $u_n \in R$ such that $f_n = d_n u_n$. Set

$$e_0 = d_0 \quad \text{and} \quad e_n = d_n(1-d_{n-1})(1-d_{n-2})\cdots(1-d_0) \quad \text{for} \quad n \in \mathbb{N},$$

and let

$$g = \sum_{n=0}^{\infty} e_n x^n \in A.$$

Since $e_0, e_1, e_2, \ldots$ are mutually orthogonal idempotents, to complete the proof it is sufficient to show that $fA = gA$.

Since R is countable-injective strongly regular, by Proposition 3.1.1 there exists a sequence $(b_n)_{n=0}^{\infty}$ of elements of R such that $e_m b_n = e_m \sigma^{-m}(f_{m+n})$ for any $m, n \in \mathbb{N} \cup \{0\}$, and since σ is constant on idempotents, we obtain

$$e_m \sigma^m(b_n) = e_m f_{m+n} \text{ for any } m, n \in \mathbb{N} \cup \{0\}. \tag{3.4.1}$$

We claim that $f = gh$, where $h = \sum_{n=0}^{\infty} b_n x^n \in A$. For this, write $gh = \sum_{n=0}^{\infty} c_n x^n$. Then applying (3.4.1) and the equality $f_n = d_n f_n$, we obtain

$$c_n = \sum_{i+j=n} e_i \sigma^i(b_j) = \sum_{i+j=n} e_i f_{i+j} = \left(\sum_{i=0}^{n} e_i\right) f_n = \left[\left(\sum_{i=0}^{n} e_i\right) d_n\right] f_n.$$

It is easy to see that $\sum_{i=0}^{n-1} e_i + \prod_{i=0}^{n-1}(1 - d_i) = 1$ for any $n \in \mathbb{N}$, and thus

$$\left(\sum_{i=0}^{n} e_i\right) d_n = \left[\sum_{i=0}^{n-1} e_i + \prod_{i=0}^{n-1}(1 - d_i)\right] d_n = 1 \cdot d_n = d_n.$$

Hence $c_n = d_n f_n = f_n$, and thus $gh = f$, which proves that $fA \subseteq gA$.

To prove the opposite inclusion, we have to show that there exists $p = \sum_{n=0}^{\infty} p_n x^n \in A$ such that $g = fp$. Equating x^n-coefficients of g and fp for all $n \in \mathbb{N} \cup \{0\}$, we obtain a system of a countable number of linear equations

$$e_n = \sum_{i+j=n} f_i \sigma^i(p_j) \quad (n = 0, 1, 2, \ldots). \tag{3.4.2}$$

Since σ is an automorphism and σ is constant on idempotents, the system (3.4.2) can be written in the equivalent form

$$e_n = \sum_{i+j=n} \sigma^{-n}(f_i) \sigma^{-j}(p_j) \quad (n = 0, 1, 2, \ldots).$$

Hence to prove the existence of $p \in A$ with $g = fp$, it suffices to show that the system

$$e_n = \sum_{i+j=n} \sigma^{-n}(f_i)x_j \quad (n = 0, 1, 2, \ldots) \tag{3.4.3}$$

has a solution in R. Since R is countable-injective strongly regular, Proposition 3.1.1 implies that the system (3.4.3) has a solution if and only if any its finite subsystem has a solution. Thus it suffices to show that for any $m \in \mathbb{N} \cup \{0\}$ there exist $y_0, y_1, \ldots, y_m \in R$ such that

$$e_n = \sum_{i+j=n} \sigma^{-n}(f_i)y_j \quad \text{for } n = 0, 1, \ldots, m. \tag{3.4.4}$$

To show that the finite system (3.4.4) has a solution, note that for any $i \in \mathbb{N} \cup \{0\}$,

$$fe_i = e_i x^i[\sigma^{-i}(u_i) + \sigma^{-i}(f_{i+1})x + \sigma^{-i}(f_{i+2})x^2 + \cdots].$$

Since $\sigma^{-i}(u_i)$ is a unit of R, [57, Proposition 2.2] implies that the power series in the brackets is a unit of A, and thus $e_i x^i = fk_i$ for some $k_i \in A$. Therefore, setting $q_m = \sum_{i=0}^m k_i \in A$, we obtain $e_0 + e_1x + e_2x^2 + \cdots + e_mx^m = fq_m$. Now it is easy to see that if $q_m = \sum_{n=0}^\infty a_nx^n$ and $y_n = \sigma^{-n}(a_n)$ for any $n \in \mathbb{N} \cup \{0\}$, then $y_0, y_1, \ldots, y_m$ is a solution of (3.4.4). □

3.5 Reduced rings of skew power series

Recall that a ring R is *reduced* if it contains no nonzero nilpotent element, i.e. $a^2 = 0$ implies $a = 0$ for any $a \in R$. Recall that an endomorphism σ of a ring R is said to be *rigid* if $a\sigma(a) \neq 0$ for every nonzero $a \in R$. This notion proved to be very useful in characterizing reduced skew polynomial rings and skew power series rings by J. Krempa in [37] and C.Y. Hong, N.K. Kim and T.K. Kwak in [29], to whom the first part of the following proposition is essentially due.

Proposition 3.5.1. *Let R be a ring and σ an endomorphism of R. Then*

(i) $R[[x;\sigma]]$ is reduced if and only if σ is rigid.

(ii) If $R[[x;\sigma]]$ is reduced, then σ is injective and idempotent-stabilizing.

Proof. (i) If $R[[x;\sigma]]$ is reduced, then for any nonzero $a \in R$ we have $a\sigma(a)x^2 = (ax)^2 \neq 0$, which proves that σ is rigid. Conversely, if σ is rigid, then it can be easily shown that R is reduced (see [29, p. 218]), and thus $R[[x;\sigma]]$ is reduced by [37, Corollary 3.5].

(ii) Obviously, σ is injective by (i). Let $e = e^2 \in R$. Since $R[[x;\sigma]]$ is reduced, idempotents of R are central, and thus

$$\sigma(e)(1-e)\sigma(\sigma(e)(1-e)) \in \sigma(e(1-e))R = \{0\}.$$

Since σ is rigid by (i), it follows that $\sigma(e)(1-e) = 0$, and thus $\sigma(e) = \sigma(e)e$. Since also $1-e$ is an idempotent, the same argument as above shows that $\sigma(1-e)e = 0$, and thus $e = \sigma(e)e = \sigma(e)$. □

3.6 Right distributive skew power series rings

Now we can formulate the following:

Theorem 3.6.1. *Let σ be an endomorphism of a ring R. Then the following conditions are equivalent:*

(1) $R[[x;\sigma]]$ is right distributive and right duo.

(2) $R[[x;\sigma]]$ is right distributive and reduced.

(3) $R[[x;\sigma]]$ is right distributive and σ is injective.

(4) $R[[x;\sigma]]$ is right Bézout and right duo.

(5) $R[[x;\sigma]]$ is right Bézout and reduced.

(6) $R[[x;\sigma]]$ is right Bézout and right quasi-duo, and σ is injective.

(7) $R[[x;\sigma]]$ is right Bézout and semicommutative, and σ is injective.

(8) $R[[x;\sigma]]$ is right Bézout and abelian, and σ is injective.

(9) All 2-generated right ideals of $R[[x;\sigma]]$ are flat, R is abelian, σ is bijective and idempotent-stabilizing.

(10) $R[[x;\sigma]]$ has weak dimension less or equal to one and $R[[x;\sigma]]$ is right duo.

(11) $R[[x;\sigma]]$ has weak dimension less or equal to one, R is abelian, σ is bijective and idempotent-stabilizing.

(12) R is countable-injective strongly regular, σ is bijective and idempotent-stabilizing.

Proof. It is easy to see that for any ring we have the following implication relations:

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & \text{right duo} \arrow[r, Rightarrow] \arrow[dd, Rightarrow] & \text{right quasi-duo} \arrow[dr, Rightarrow] & \\
(*) & & & \text{Dedekind-finite} \\
\text{reduced} \arrow[r, Rightarrow] & \text{semicommutative} \arrow[r, Rightarrow] & \text{abelian} \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

From Theorem 3.1.10 we obtain (3) $\Leftrightarrow$ (12), and by Theorem 3.1.11 we have (11) $\Leftrightarrow$ (9) $\Leftrightarrow$ (12). Thus the conditions (3), (9), (11) and (12) are equivalent.

We continue the proof by showing first that

$$(12) \Rightarrow R[[x;\sigma]] \text{ is right duo.} \tag{3.6.1}$$

To prove (3.6.1), it suffices to show that for the ring $A = R[[x;\sigma]]$ and any $f \in A$ we have $Af \subseteq fA$. By Proposition 3.4.1 we can assume that $f = \sum_{n=0}^{\infty} e_n x^n$, where $(e_n)_{n=0}^{\infty}$ is a sequence of mutually orthogonal idempotents of R, and we have to show that $gf \in fA$ for any $g = \sum_{n=0}^{\infty} g_n x^n \in A$. Since R is countable-injective strongly regular, by Proposition 3.1.1 there exists a sequence $(t_m)_{m=0}^{\infty}$ of elements of R such that for any $m, n \in \mathbb{N} \cup \{0\}$ we have $e_n t_m = e_n \sigma^{-n}(g_m)$, or equivalently, $e_n \sigma^n(t_m) = e_n g_m$. Set $h = \sum_{m=0}^{\infty} t_m x^m \in A$. Now it is easy to verify that $gf = fh$, which completes the proof of (3.6.1).

Since (3) $\Leftrightarrow$ (12), it follows from (3.6.1) that (3) $\Rightarrow$ (1). Since (1) $\Rightarrow$ (3) is a direct consequence of Proposition 3.3.1, we obtain (3) $\Leftrightarrow$ (1).

Using again the equivalence (3) $\Leftrightarrow$ (12), we deduce from Proposition 3.5.1(i) that (3) $\Rightarrow$ (2). Since the opposite implication is a consequence of Proposition 3.5.1(ii), (3) $\Leftrightarrow$ (2) follows.

At this point we know that the conditions (1), (2), (3), (9), (11) and (12) are equivalent. Now we show that they are equivalent to the conditions (4) through (8).

By Theorem 3.1.10, (3) implies that $R[[x;\sigma]]$ is right Bézout, and since (3) $\Leftrightarrow$ (12), the implication (3) $\Rightarrow$ (4) follows from (3.6.1). Hence applying the implication chart $(*)$ and Proposition 3.3.1, we obtain (3) $\Rightarrow$ (4) $\Rightarrow$ (6). Furthermore, since all right quasi-duo right Bézout rings are right distributive (see Proposition 3.1.3(ii)), we deduce that (6) $\Rightarrow$ (3). Moreover, since we already know that (3) $\Leftrightarrow$ (2) and (3) $\Rightarrow$ (4), it follows that (3) $\Rightarrow$ (5). Hence, applying $(*)$ and Proposition 3.5.1(ii), we obtain (3) $\Rightarrow$ (5) $\Rightarrow$ (7) $\Rightarrow$ (8). Since obviously (8) implies that R is abelian, and thus Dedekind-finite, it follows from Proposition 3.2.1 that R is von Neumann regular. Thus for every element $r \in R$, there exists $x \in R$ such that $r = rxr$. Then $rx = rxrx$ is idempotent in R. So, since R is abelian $r = rxr = r^2x$ and it follows that R is strongly regular. Hence R is semicommutative, and from Theorem 3.1.10 we obtain (8) $\Rightarrow$ (3). Thus we have shown that the conditions (4)–(8) are equivalent to (3).

By the above, the conditions (1)–(9), (11) and (12) are equivalent. By $(*)$, (10) implies that $R[[x;\sigma]]$ is abelian, and thus so is R. Hence using Proposition 3.3.1, we obtain (10) $\Rightarrow$ (11). On the other hand, we know that (11) $\Leftrightarrow$ (12), and thus it follows from (3.6.1) that (11) $\Rightarrow$ (10), proving that (11) $\Leftrightarrow$ (10). Hence the conditions (1)–(12) are equivalent. $\square$

Proposition 3.6.2. *All conditions which appear in the formulation of Theorem 3.6.1 imply the left analogues of* (1)–(10), *and in particular, if any of these conditions holds, then* $R[[x;\sigma]]$ *is duo. Moreover, if* σ *is an automorphism, then these conditions are equivalent to the left analogues of* (1)–(10).

Proof. Assume that one of the conditions of above theorem is satisfied. Then (12) is satisfied, and thus R is countable-injective strongly regular, σ is bijective and idempotent-stabilizing. Hence R^{op}, the opposite ring of R, is countable-injective strongly regular, σ^{-1} is an automorphism of R^{op} and $\sigma^{-1}(e) = e$ for any $e = e^2 \in R$. Thus (12) holds for the ring R^{op} and its automorphism σ^{-1}. Hence, again by the first part of the theorem, any of the conditions (1)–(10) holds for the ring R^{op} and its automorphism σ^{-1}, i.e., for the skew power series ring $R^{\mathrm{op}}[[x;\sigma^{-1}]]$. Since $R^{\mathrm{op}}[[x;\sigma^{-1}]]$ is isomorphic to the opposite ring $R[[x;\sigma]]^{\mathrm{op}}$ of the ring $R[[x;\sigma]]$, and the conditions (1)–(10) are "right-sided", it follows that $R[[x;\sigma]]$ and R satisfy the "left-sided" versions of (1)–(10), that is, the left analogues of (1)–(10).

To complete the proof it suffices to show that if σ is bijective and for some $i \in \{1, 2, \ldots, 10\}$ the left analogue of (i) is satisfied, then (12) is satisfied. But if the left analogue of (i) is satisfied, then from the isomorphism $R[[x;\sigma]]^{\mathrm{op}} \simeq R^{\mathrm{op}}[[x;\sigma^{-1}]]$ it follows that (i) is satisfied for $R^{\mathrm{op}}[[x;\sigma^{-1}]]$, R^{op} and σ^{-1}. Hence, by Theorem 3.6.1, (12) is satisfied for R^{op} and σ^{-1}. Thus (12) is also satisfied for R and σ. $\square$

The following example shows that in the Proposition 3.6.2 the assumption that σ is bijective is essential.

Example 3.6.3. Let $K = F(y_1, y_2, y_3, \ldots)$ be the rational function field in infinitely many variables y_n over a field F, and let σ be the endomorphism of K defined by $\sigma(y_n) = y_{n+1}$ for any $n \in \mathbb{N}$, and $\sigma(f) = f$ for any $f \in F$. Then σ is an injective endomorphism of K that is not an automorphism. Set $A = K[[x;\sigma]]$. If $g \in A \setminus \{0\}$, then g can be written in the form $g = (k_0 + k_1x + k_2x^2 + \cdots)x^n$ with invertible k_0, and thus $Ag = Ax^n$. Now it easily follows that left ideals of A are totally ordered by set inclusion, and thus A is left distributive. Hence the left analogue of (3) is satisfied, whereas obviously (12) is not satisfied. $\square$

As the final part of the section we want to consider projective modules and start considerations which together with Proposition 4.2.16 will show that skew power series rings and generalized power series rings (which we will introduce in the next chapter) behave differently regarding projectivity.

Definition 3.6.4. A right R-module P is called *projective* if for every right R-modules A and B, surjective R-homomorphism $h : A \to B$, and R-homomorphism $f : P \to B$ there exists an R-homomorhism $\alpha : P \to A$ such that $h\alpha = f$.

Recall that if every finitely generated ideal (resp. every principal right ideal) of a ring R is projective as a right R-module, then R is called *right semihereditary* (resp. *right Rickartian*)

Remark 3.6.5. It is well known (e.g. see [41, Exercise 2.2, page 55]) that a ring R is right Rickartian if and only if the right annihilator of any element of R is generated (as a right ideal) by an idempotent of R.

The following result (see [14, Theorem 1]) is interesting in the context of Theorem 3.6.1 but also we will use it later on during considaration about generalized power series rings.

Theorem 3.6.6. *If a ring R is right duo, then R is right semihereditary if and only if R is right Rickartian and R has weak dimension less or equal to one.*

Having proved Theorem 3.6.1 and keeping in mind the above theorem, one could ask if this is possible to add to the equivalent conditions in Theorem 3.6.1 a new one. Namely, the condition "$R[[x;\sigma]]$ is right semihereditary and right duo." The following example removes the possibility (cf. [80, 4.89]).

Example 3.6.7. There exists a ring R such that the ring $R[[x]]$ satisfies all conditions which appear in Theorem 3.6.1, but $R[[x]]$ is not right semihereditary.

Proof. Let $R = \prod_{i=1}^{\infty} F_i / \bigoplus_{i=1}^{\infty} F_i$ where for every i, $F_i = \mathbb{Z}/2\mathbb{Z}$ is the ring of integers modulo 2. It is obvious that R is commutative and since for every $r \in R$, $r = r^2$, the ring R is strongly regular.

Using Proposition 3.1.1(6) it is easy to see that the ring $\prod_{i=1}^{\infty} F_i$ is countable-injective. Now Proposition 3.1.1(5) implies that R is countable-injective. Hence the ring $R[[x]]$ satisfies all conditions which appear in Theorem 3.6.1.

For every $a \in \prod_{i=1}^{\infty} F_i$ let $d(a) = \{j \in \mathbb{N} : \pi_j(a) = 1\}$, where π_j denotes the natural projection. If $a \in \prod_{i=1}^{\infty} F_i$, then the image of a in R will be denoted by $\overline{a}$.

Now we would like to show that $R[[x]]$ is not semihereditary. For that, let $(p_n)_{n\in\mathbb{N}}$ be a sequence of prime numbers, such that for $i \neq j$ we have $p_i \neq p_j$. Moreover, for every positive integer i let $N_i = \{p_i^k : k \in \mathbb{N}\}$ and w_i be such element of $\prod_{i=1}^{\infty} F_i$ that $\pi_n(w_i) = 1$ if $n \in N_i$ and $\pi_n(w_i) = 0$ otherwise. It is obvious that $w_i \neq w_j$ and $w_i w_j = 0$ for $i \neq j$.

Now let $f = \sum_{i=1}^{\infty} \overline{w_i} x^{i-1} \in R[[x]]$, and let us suppose that $R[[x]]$ is semihereditary. Then $fR[[x]]$ is projective, so Remark 3.6.5 implies that $r_{R[[x]]}(f) = eR[[x]]$, for some idempotent $e \in R[[x]]$.

Since the ring $R[[x]]$ satisfies all conditions which appear in Theorem 3.6.1, by Proposition 3.4.1, $r_{R[[x]]}(f) = (\sum_{i=1}^{\infty} \overline{e_i} x^{i-1})R[[x]]$ where $e_1, e_2, \dots \in \prod_{i=1}^{\infty} F_i$ and $\overline{e_1}, \overline{e_2}, \dots$ are mutually orthogonal idempotents of R.

Since $f \cdot \sum_{i=1}^{\infty} \overline{e_i} x^{i-1} = 0$, for every positive integer n we have

$$0 = \overline{w_n} f(\overline{e_1} + \overline{e_2}x + \overline{e_3}x^2 + \cdots)\overline{e_1} = \overline{w_n e_1} x^{n-1},$$

what implies that $w_n e_1 \in \bigoplus_{i=1}^{\infty} F_i$. Thus the set $d(e_1) \cap N_n$ is finite for every n. Let $(q_n)_{n=1}^{\infty}$ be a sequence of positive integers such that for every $n \in \mathbb{N}$, $p_n^{q_n} \in N_n \backslash d(e_1)$.

Now, let us consider the element $a \in \prod_{i=1}^{\infty} F_i$ such that $\pi_m(a) = 1$ if $m = p_n^{q_n}$ for some $n \in \mathbb{N}$ and $\pi_m(a) = 0$ otherwise. Then $\overline{a} \neq 0$ and $\overline{a} \cdot \overline{e_1} = 0$.

It is obvious that for every positive integer n, $\overline{w_n} \cdot \overline{a} = 0$. So $\overline{a} \in r_{R[[x]]}(f) = (\sum_{i=1}^{\infty} \overline{e_i} x^{i-1})R[[x]]$. Hence $\overline{a} = (\sum_{i=1}^{\infty} \overline{e_i} x^{i-1})g$ for some $g \in R[[x]]$. Since $\overline{a} = \overline{a}^2$, we have $\overline{a} = \overline{a} \cdot (\sum_{i=1}^{\infty} \overline{e_i} x^{i-1})g$. Equating constant terms in the last equation we get $\overline{a} = \overline{a} \cdot \overline{e_1} \cdot v = 0$ for some $v \in R$, a contradiction. Thus $R[[x]]$ is not semihereditary as we claimed. □

3.7 Right Gaussian skew power series rings

To prove the main result of this chapter we need the following (cf. [80, 4.58]):

Lemma 3.7.1. *For any ring R and an endomorphim σ of R, the following conditions are equivalent:*

(1) R is strongly regular and σ is idempotent-stabilizing.

(2) σ is injective and for any $a \in R$ there exists $b \in R$ such that $\sigma(a) = \sigma(a)ab$.

Proof. (1) $\Rightarrow$ (2) Since R is strongly regular, each element of R is a product of a unit and an idempotent, and thus (1) implies that σ is injective. Moreover, for any $a \in R$ there exists $b \in R$ such that $a = a^2b$ and $ab = (ab)^2$, and it follows from (1) that $\sigma(a) = \sigma(a)ab$.

(2) $\Rightarrow$ (1) We first show that R is a reduced ring. For, let $a \in R$ and $a^2 = 0$. Then $\sigma(\sigma(a)a)\sigma(a)a = \sigma(\sigma(a)a^2)a = 0$, and since by (2) we have $\sigma(\sigma(a)a) \in \sigma(\sigma(a)a)\sigma(a)aR$, it follows that $\sigma(\sigma(a)a) = 0$. Hence $\sigma(a)a = 0$ by the injectivity of σ, and we deduce from (2) that $\sigma(a) = 0$, which implies that $a = 0$, as desired.

Next we show that the ring R is strongly regular. By (2), for any $a \in R$ there exists $b \in R$ such that $\sigma(a) = \sigma(a)ab$. Hence $\sigma(a)c = 0$ for $c = 1 - ab$, and since R is reduced, it follows from (2) that $\sigma(ac) \in \sigma(ac)acR \subseteq \sigma(a)RcR = \{0\}$. Thus $ac = 0$ by the injectivity of σ, which shows that $a = a^2b$.

Finally we prove that σ is idempotent-stabilizing. Let $e = e^2 \in R$. By (2) there exist $c, d \in R$ such that $\sigma(e) = \sigma(e)ec$ and $\sigma(1-e) = \sigma(1-e)(1-e)d$. Thus $1 = \sigma(e) + \sigma(1-e) = \sigma(e)ec + \sigma(1-e)(1-e)d$, and since R is reduced, e is central and $e = \sigma(e)ec = \sigma(e)$ follows. □

Theorem 3.7.2. *Let σ be an endomorphism of a ring R. Then the following conditions are equivalent:*

(1) $R[[x;\sigma]]$ is right Gaussian.

(2) $R[[x;\sigma]]$ is right duo right distributive.

(3) $R[[x;\sigma]]$ is reduced right distributive.

(4) $R[[x;\sigma]]$ is right distributive and σ is injective.

(5) $R[[x;\sigma]]$ is right duo of weak dimension less than or equal to one.

(6) $R[[x;\sigma]]$ is right duo right Bézout.

(7) $R[[x;\sigma]]$ is reduced right Bézout.

(8) $R[[x;\sigma]]$ is right quasi-duo right Bézout and σ is injective.

(9) $R[[x;\sigma]]$ is semicommutative right Bézout and σ is injective.

(10) R is countable-injective strongly regular, and σ is bijective and idempotent-stabilizing.

Proof. It follows immediately from Theorem 3.6.1 that conditions (2)–(10) are equivalent.

(2) $\Rightarrow$ (1) is a direct consequence of Theorem 2.2.11.

(1) $\Rightarrow$ (10) Assume that the ring $A = R[[x;\sigma]]$ is right Gaussian. Then A is right duo by Lemma 2.2.5, and thus Proposition 3.3.1 implies that σ is bijective and idempotent-stabilizing.

By Lemma 3.7.1, to prove that R is strongly regular, it suffices to show that $\sigma(a) \in \sigma(a)aR$ for any $a \in R$. To get the latter we observe that since A is right Gaussian and in the polynomial ring $A[X]$ we have $(\sigma(a) + xX)(a - xX) = \sigma(a)a - x^2X^2$, there exist $f, g \in A$ such that

$$\sigma(a)x = \sigma(a)af + x^2g. \tag{3.7.1}$$

Equating x-coefficients in (3.7.1), we obtain $\sigma(a) \in \sigma(a)aR$, as desired.

We already know that R is strongly regular, and thus by Proposition 3.1.1, to prove that R is countable-injective it suffices to show that for any countable set $\{e_1, e_2, \dots\}$ of mutually orthogonal idempotents of R and any countable set $\{a_1, a_2, \dots\}$ of elements of R there exists $c \in R$ such that $e_n a_n = e_n c$ for all $n \in \mathbb{N}$. To prove the condition, for any n we set $b_n = \sigma^{4n-4}(a_n)$, and we observe that since σ is idempotent-stabilizing, in $A[X]$ we have

$$\left[\sum_{n=1}^{\infty} e_n x^{2n-2} + \left(\sum_{n=1}^{\infty} e_n b_n x^{2n-1}\right) X\right] \cdot \left[\sum_{n=1}^{\infty} e_n x^{2n-2} - \left(\sum_{n=1}^{\infty} e_n \sigma^{2-2n}(b_n) x^{2n-1}\right) X\right] =$$

$$= \sum_{n=1}^{\infty} e_n x^{4n-4} - \left(\sum_{n=1}^{\infty} e_n b_n \sigma(b_n) x^{4n-2}\right) X^2.$$

Since A is right Gaussian, the above equality implies that there exist $f, g \in A$ with

$$\sum_{n=1}^{\infty} e_n b_n x^{4n-3} = \left(\sum_{n=1}^{\infty} e_n x^{2n-2}\right) \cdot \left(\sum_{n=1}^{\infty} e_n \sigma^{2-2n}(b_n) x^{2n-1}\right) =$$

$$= \left(\sum_{n=1}^{\infty} e_n x^{4n-4}\right) f + \left(\sum_{n=1}^{\infty} e_n b_n \sigma(b_n) x^{4n-2}\right) g.$$

For any $n \in \mathbb{N}$, multiplying the above equation by e_n from the left, we get

$$e_n b_n x^{4n-3} = e_n x^{4n-4} f + e_n b_n \sigma(b_n) x^{4n-2} g. \tag{3.7.2}$$

Equating x^{4n-3}-coefficients in (3.7.2), we obtain $e_n b_n = e_n \sigma^{4n-4}(c)$, where c is the x-coefficient of f. Since $b_n = \sigma^{4n-4}(a_n)$, and σ is bijective and idempotent-stabilizing, it follows that $e_n a_n = e_n c$ for all n, proving that R is countable-injective. □

As a direct consequence of Theorem 3.7.2 we obtain the characterization of a right Gaussian power series ring $R[[x]]$.

Corollary 3.7.3. *For any ring R the following conditions are equivalent:*

(1) $R[[x]]$ is right Gaussian.

(2) $R[[x]]$ is right duo right distributive.

(3) $R[[x]]$ is reduced right distributive.

(4) $R[[x]]$ is right distributive.

(5) $R[[x]]$ is right duo of weak dimension less than or equal to one.

(6) $R[[x]]$ is right duo right Bézout.

(7) $R[[x]]$ is reduced right Bézout.

(8) $R[[x]]$ is right quasi-duo right Bézout.

(9) $R[[x]]$ is semicommutative right Bézout.

(10) R is countable-injective strongly regular.

A particular case of the above corollary is the following result of Anderson and Camillo (see [2, Theorem 17]) characterizing commutative Gaussian power series rings.

Corollary 3.7.4. *For any commutative ring R the following conditions are equivalent:*

(1) $R[[x]]$ is Gaussian.

(2) $R[[x]]$ is (reduced) distributive.

(3) $R[[x]]$ has weak dimension one.

(4) $R[[x]]$ is Bézout.

(5) R is von Neumann regular and countable-injective.

At this moment we are ready to show that as contrasted with the direct product (see Corollary 2.2.3), there exists a ring which is subdirect product of right Gaussian rings but is not right Gaussian itself (see [51, Example 6]).

Example 3.7.5. Let D be a division ring of characteristic equal to 0. We will consider the ring

$$R = \{(d_1, d_2, \dots) \in \prod_{i \in \mathbb{N}} D_i : D_i = D, \text{ there exists } n \text{ such that } d_n = d_{n+1} = \dots\}.$$

It is clear that R is strongly regular. Let for every $i \in \mathbb{N} \cup \{0\}$, $a_i = (d_1, d_2, \dots)$, $e_i = (c_1, c_2, \dots)$ be such elements of R that $d_{i+1} = i + 1$, $c_{i+1} = 1$ and $d_{j+1} = c_{j+1} = 0$ for $j \neq i$. Then every e_i is idempotent of R and it is obvious that there does not exist $d \in R$ such that $a_i e_i = d e_i$ for every $i \in \mathbb{N}$. Thus by Proposition 3.1.1 R is not countable-injective. Hence Corollary 3.7.3 implies that $R[[x]]$ is not right Gaussian.

On the other hand it is easy to see that for every $i \in \mathbb{N}$ the set $m_i = \{(d_1, d_2, \dots) \in R : d_i = 0\}$ is the maximal ideal of R, $R/m_i \cong D_i = D$ and $m_i[[x]] = \{f = a_0 + a_1 x + \dots \in R[[x]] : a_0, a_1, \dots \in m_i\}$ is an ideal of $R[[x]]$. Since $R[[x]]/m_i[[x]] \cong (R/m_i)[[x]] \cong D_i[[x]] = D[[x]]$ and $\bigcap_{i \in \mathbb{N}} m_i[[x]] = \{0\}$ it follows that $R[[x]]$ can be represented as a subdirect product of $\{D_i[[x]] : i \in \mathbb{N}\}$. By Corollary 3.7.3 the ring $D_i[[x]] = D[[x]]$ is right Gaussian. Thus $R[[x]]$ is subdirect product of right Gaussian rings but is not right Gaussian itself. $\square$

Chapter 4

Right Gaussian skew generalized power series rings

This Chapter is based on:

• R. Mazurek, M. Ziembowski, *Weak dimension and right distributivity of skew generalized power series rings*, to appear in Journal of the Mathematical Society of Japan.

and on part of:

• R. Mazurek, M. Ziembowski, *Right Gaussian rings and skew power series rings*, submitted.

The aim of the chapter is to extend Theorem 3.7.2 to the skew generalized power series rings. The skew generalized power series construction, introduced in [57], embraces a wide range of classical ring-theoretic extensions, including skew polynomial rings, skew power series rings, skew Laurent polynomial rings, skew group rings, Mal'cev-Neumann Laurent series rings, the "untwisted" versions of all of these, and the "untwisted" rings of generalized power series (see [69] for the definition of the last class of rings).

4.1 Construction of skew generalized power series rings

In order to construct the skew generalized power series ring we need some definitions and facts.

When we consider an ordering relation $\leqslant$ on a set S, then the word "order" means a partial ordering unless otherwise stated. An order $\leqslant$ is *total* (respectively, *trivial*) if any two different elements of S are comparable (respectively, incomparable) with respect to $\leqslant$.

Let $(S, \leqslant)$ be an ordered set. Then $(S, \leqslant)$ is called *artinian* if every strictly decreasing sequence of elements of S is finite, and $(S, \leqslant)$ is called *narrow* if every subset of pairwise order-incomparable elements of S is finite. Thus $(S, \leqslant)$ is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements. A set which is artinian and narrow is also called a well-partially-ordered set (see [38]). Such sets are characterized in the following proposition.

Proposition 4.1.1. ([15, Corollary 1]) *Let $(S, \leqslant)$ be an ordered set. The following conditions are equivalent:*

(1) *$(S, \leqslant)$ is artinian and narrow.*

(2) *For any sequence $(s_n)_{n\in\mathbb{N}}$ of elements of S there exist indices $n_1 < n_2 < n_3 < \ldots$ such that $s_{n_1} \leqslant s_{n_2} \leqslant s_{n_3} \leqslant \ldots$.*

(3) *For any sequence $(s_n)_{n\in\mathbb{N}}$ of elements of S there exist indices $i < j$ such that $s_i \leqslant s_j$.*

Clearly, the union of a finite family of artinian and narrow subsets of an ordered set as well as any subset of an artinian and narrow set are again artinian and narrow.

Let $(S, \cdot)$ be a monoid with an identity element 1 and let $\leqslant$ be an order relation on S. We say that $(S, \cdot, \leqslant)$ is an *ordered monoid* if for any $s_1, s_2, t \in S$, $s_1 \leqslant s_2$ implies $s_1 t \leqslant s_2 t$ and $t s_1 \leqslant t s_2$. Moreover, if $s_1 < s_2$ implies $s_1 t < s_2 t$ and

$ts_1 < ts_2$, then $(S, \cdot, \leqslant)$ is said to be a *strictly ordered monoid.* If $(S, \cdot)$ is a group, then we will say that $(S, \cdot, \leqslant)$ is an *ordered group.*

If $(S, \cdot)$ is a monoid, $n \in \mathbb{N}$ and $T_1, T_2, \ldots, T_n, T$ are nonempty subsets of S, then $T_1 T_2 \cdots T_n$ (respectively, T^n) will denote the set of all products $t_1 t_2 \cdots t_n$ with $t_i \in T_i$ (respectively, $t_i \in T$) for any $i \in \{1, ..., n\}$.

Using Proposition 4.1.1 it is not too hard to prove the following (see [57, Proposition 1.2]):

Proposition 4.1.2. *Let $(S, \cdot, \leqslant)$ be an ordered monoid and $T_1, T_2, \ldots, T_n$ $(n \geqslant 1)$ artinian and narrow subsets of S. Then*

(i) *The set $T_1 T_2 \cdots T_n$ is artinian and narrow.*

(ii) *If $(S, \cdot, \leqslant)$ is strictly ordered, then for any $s \in S$ the set $\{(t_1, t_2, \ldots, t_n) \in T_1 \times T_2 \times \cdots \times T_n : s = t_1 t_2 \cdots t_n\}$ is finite.*

Given a ring R, a strictly ordered monoid $(S, \leqslant)$ and a monoid homomorphism ω: $S \to \mathrm{End}(R)$ (for every $s \in S$, instead $\omega(s)$ we will put down ω_s), consider the set A of all maps $f : S \to R$ whose support $\mathrm{supp}(f) = \{s \in S : f(s) \neq 0\}$ is artinian and narrow. If $f, g \in A$ and $s \in S$, by Proposition 4.1.2 the set

$$\mathrm{X}_s(f, g) = \{(x, y) \in \mathrm{supp}(f) \times \mathrm{supp}(g) : s = xy\}$$

is finite. Thus one can define the product fg: $S \to R$ of $f, g \in A$ as follows:

$$(fg)(s) = \sum_{(x,y) \in \mathrm{X}_s(f,g)} f(x)\, \omega_x(g(y)) \quad \text{for any} \quad s \in S$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, A becomes a ring, called the *ring of skew generalized power series with coefficients in R and exponents in S*, and is denoted by $R[[S, \omega]]$ (or $R[[S, \omega, \leqslant]]$ to indicate the order $\leqslant$). If we consider the ring $R[[S, 1]]$, then it means that ω is a such monoid homomorphism ω: $S \to \mathrm{End}(R)$, for which $\omega_s = id_R$ for every $s \in S$.

From now on we will use the symbol 1 to denote the identity elements of the monoid S, the ring R and the ring $R[[S,\omega]]$. To each $r \in R$ and $s \in S$, we associate elements $\mathsf{c}_r, \mathsf{e}_s \in R[[S,\omega]]$ defined by

$$\mathsf{c}_r(x) = \begin{cases} r & \text{if } x = 1 \\ 0 & \text{if } x \in S \setminus \{1\}, \end{cases} \qquad \mathsf{e}_s(x) = \begin{cases} 1 & \text{if } x = s \\ 0 & \text{if } x \in S \setminus \{s\}. \end{cases}$$

It is clear that $r \mapsto \mathsf{c}_r$ is a ring embedding of R into $R[[S,\omega]]$ and $s \mapsto \mathsf{e}_s$ is a monoid embedding of S into the multiplicative monoid of the ring $R[[S,\omega]]$. Furthermore, we have $\mathsf{e}_s\mathsf{c}_r = \mathsf{c}_{\omega_s(r)}\mathsf{e}_s$ for any $r \in R$ and $s \in S$.

The construction of generalized power series rings generalizes some classical ring constructions such as:

(1) polynomial rings ($S = \mathbb{N} \cup \{0\}$ with usual addition, and trivial $\leqslant$ and ω),

(2) monoid rings (trivial $\leqslant$ and ω),

(3) skew polynomial rings ($S = \mathbb{N} \cup \{0\}$ with usual addition and trivial $\leqslant$),

(4) skew Laurent polynomial rings ($S = \mathbb{Z}$ with usual addition and trivial $\leqslant$),

(5) skew monoid rings (trivial $\leqslant$),

(6) skew power series rings ($S = \mathbb{N} \cup \{0\}$ with usual addition and usual order),

(7) skew Laurent series rings ($S = \mathbb{Z}$ with usual addition and usual order),

(8) the Mal'cev-Neumann construction ($(S, \cdot, \leqslant)$ a totally ordered group and trivial ω; see [13, p. 528]),

(9) the Mal'cev-Neumann construction of twisted Laurent series rings ($(S, \cdot, \leqslant)$ a totally ordered group; see [39, p. 242]),

(10) generalized power series rings (trivial ω; see [69, Section 4]).

Recall that an ordered monoid $(S, \leqslant)$ is *positively ordered* if $s \geqslant 1$ for any $s \in S$. An obvious example of such a monoid is $S_0 = \mathbb{N} \cup \{0\}$ under addition, with its natural linear order. It is clear that if σ is an endomorphism of a ring R, then the map $\omega : S_0 \to \mathrm{End}(R)$ given by $\omega(n) = \sigma^n$ for any $n \in S_0$, is a monoid homomorphism, and the ring $R[[S_0, \omega]]$ is isomorphic to the skew power series ring $R[[x; \sigma]]$. Hence, skew power series rings can be considered as a special case of skew generalized power series rings with positively ordered exponents.

4.2 Right distributive skew generalized power series rings

In this section we study relations between the weak dimension, the right distributivity, and the right Bézout condition of skew generalized power series rings $R[[S, \omega]]$ with positively ordered exponents.

The following result will allow us to apply in these studies the important connection given in Proposition 3.1.3(ii).

Lemma 4.2.1. *Let R be a ring, $(S, \leqslant)$ a positively strictly ordered monoid, $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism, and let $A = R[[S, \omega]]$. Then*

$$J(A) = \{f \in A : f(1) \in J(R)\}$$

and $A/J(A) \cong R/J(R)$.

Proof. Since S is positivelly ordered, $(fg)(1) = f(1)g(1)$ for any $f, g \in A$, and it follows that the map $\varphi : A \to R/J(R)$, $\varphi(f) = f(1) + J(R)$, is a ring epimorphism with $\ker \varphi = \{f \in A : f(1) \in J(R)\}$. Hence to complete the proof, it suffices to show that $\ker \varphi = J(A)$. If $f \in \ker \varphi$, then $f(1) \in J(R)$ and thus for any $g \in A$ we have $(1 - gf)(1) = 1 - g(1)f(1) \in 1 + J(R) \subseteq U(R)$. Hence [57, Proposition 2.2] implies that $1 - gf \in U(A)$, and thus $f \in J(A)$ by [39, Lemma 4.1]. Therefore, $\ker \varphi \subseteq J(A)$, and since $A/\ker \varphi \cong R/J(R)$ is Jacobson semisimple, from [39, Proposition 4.6] we deduce that $\ker \varphi = J(A)$. $\square$

We will often use the following property of right ideals of a ring which are flat as right R-modules.

Lemma 4.2.2. (see [80, 4.23]) *Let a, b, c, d be elements of a ring R such that $ab = cd$ and $aR + cR$ is a flat as a right R-module. Then there exist $f, g, h, k \in R$ such that $af = cg$, $(1-f)b = hd$, $ah = ck$ and $(1-k)d = gb$.*

Recall that an element s of a monoid $(S, \cdot)$ is *right cancellative* (resp. *left cancellative*) if $xs = ys$ (resp. $sx = sy$) implies $x = y$ for any $x, y \in S$. If every element of S is right cancellative (resp. left cancellative), then we will say that S is a *right cancellative monoid* (resp. *left cancellative monoid*). Right and left cancellative monoids are called *cancellative*.

Now, we can prove the following:

Lemma 4.2.3. *Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that the ring $R[[S, \omega]]$ has weak dimension less than or equal to one. Then S is cancellative if and only if for any $s, t, w \in S$, $stw = sw$ implies $t = 1$.*

Proof. The "only if" part is obvious. To prove the "if" part, assume that

$$stw = sw \text{ implies } t = 1 \text{ for all } s, t, w \in S, \tag{4.2.1}$$

and consider any $s, u, v \in S$ with $su = sv$. Then in the ring $A = R[[S, \omega]]$ we have $\mathsf{e}_s\mathsf{e}_u = \mathsf{e}_s\mathsf{e}_v$, and thus by Lemma 4.2.2 there exist $f, g, h, k \in A$ such that

$$\mathsf{e}_s f = \mathsf{e}_s g, \;\; (1-f)\mathsf{e}_u = h\mathsf{e}_v, \;\; \mathsf{e}_s h = \mathsf{e}_s k \text{ and } (1-k)\mathsf{e}_v = g\mathsf{e}_u. \tag{4.2.2}$$

Suppose that $u \notin Sv$. Then (4.2.1) and (4.2.2) imply that

$$0 = (h\mathsf{e}_v)(u) = ((1-f)\mathsf{e}_u)(u) = 1 - f(1)$$

and

$$0 = ((1-k)\mathsf{e}_v)(u) = (g\mathsf{e}_u)(u) = g(1).$$

Thus from the first part of (4.2.2) we obtain

$$1 = \omega_s(1) = \omega_s(f(1)) = (\mathsf{e}_s f)(s) = (\mathsf{e}_s g)(s) = \omega_s(g(1)) = \omega_s(0) = 0,$$

and this contradiction shows that $u = tv$ for some $t \in S$. Hence $stv = su = sv$, and (4.2.1) implies that $t = 1$. Thus $u = v$, which proves that S is left cancellative. Similarly one can show that S is right cancellative. $\square$

Since every idempotent of a right distributive ring is central (see [72, Corollary 2 of Proposition 1.1]), the following lemma implies that for any strictly ordered monoid $(S, \leqslant)$, if the ring $R[[S,\omega]]$ is right distributive, then ω_s is idempotent-stabilizing for any $s \in S$.

Lemma 4.2.4. *Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that the ring $R[[S,\omega]]$ is abelian. Then for any $s \in S$, ω_s is idempotent-stabilizing.*

Proof. Set $A = R[[S,\omega]]$ and consider any $s \in S$ and $e = e^2 \in R$. Then $\mathsf{c}_e = \mathsf{c}_e^2$ in A, and since A is abelian, we obtain $\mathsf{c}_e \mathsf{e}_s = \mathsf{e}_s \mathsf{c}_e = \mathsf{c}_{\omega_s(e)} \mathsf{e}_s$. Hence $\omega_s(e) = e$. $\square$

Corollary 4.2.5. *Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that the ring $R[[S,\omega]]$ is right distributive. Then for any $s \in S$, ω_s is idempotent-stabilizing.*

The following lemma implies, in particular, that for any positively strictly ordered monoid $(S, \leqslant)$, if the ring $R[[S,\omega]]$ is right distributive, then S is left cancellative.

Lemma 4.2.6. *Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that the ring $R[[S,\omega]]$ is right distributive. Then S is left cancellative if and only if for any $s, t \in S$, $st = s$ implies $t = 1$.*

Proof. The "only if" part is clear. For the "if" part, assume that

$$st = s \ \text{ implies } \ t = 1 \ \text{ for any } \ s, t \in S, \tag{4.2.3}$$

and consider any $s, u, v \in S$ with $su = sv$. Then in the ring $A = R[[S, \omega]]$ we have $\mathsf{e}_s\mathsf{e}_u = \mathsf{e}_s\mathsf{e}_v$, and thus by Theorem 2.1.2 there exist $f, g, h, k \in A$ such that

$$f + g = 1, \quad \mathsf{e}_u f = \mathsf{e}_v h \quad \text{and} \quad \mathsf{e}_v g = \mathsf{e}_u k. \tag{4.2.4}$$

Suppose that $u \notin vS$ and $v \notin uS$. Then (4.2.3) and (4.2.4) imply that $\omega_u(f(1)) = (\mathsf{e}_u f)(u) = (\mathsf{e}_v h)(u) = 0$, and analogously one shows that $\omega_v(g(1)) = 0$. Since $su = sv$, it follows that $\omega_{su}(f(1)) = \omega_{su}(g(1)) = 0$. Now from the first part of (4.2.4) we obtain

$$1 = \omega_{su}(1) = \omega_{su}((f+g)(1)) = \omega_{su}(f(1)) + \omega_{su}(g(1)) = 0 + 0 = 0,$$

and this contradiction shows that $u \in vS$ or $v \in uS$. In the first case $u = vt$ for some $t \in S$. Hence $sv = su = svt$, and (4.2.3) implies that $t = 1$, which leads to $u = v$, as desired. The case when $v \in uS$ follows similarly. □

Let S be a monoid. Recall that S is a *right chain monoid* if the right ideals of S are totally ordered by set inclusion ([17]), i.e. for any $s, t \in S$ we have $sS \subseteq tS$ or $tS \subseteq sS$. Recall also that S is said to be *right duo* if all right ideals of S are two-sided ideals, i.e. $St \subseteq tS$ for any $t \in S$.

Lemma 4.2.7. *Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that ω_s is injective for any $s \in S$. Then S is a right chain monoid in each of the following cases:*

(i) If $R[[S, \omega]]$ is right distributive and S is left cancellative.

(ii) If $R[[S, \omega]]$ has weak dimension less than or equal to one, and S is cancellative and right duo.

Proof. (i) Assume that S is not a right chain monoid. Then there exist $s, t \in S$ with $sS \nsubseteq tS$ and $tS \nsubseteq sS$. Since the ring $R[[S, \omega]]$ is right distributive, by Theorem 2.1.2 for some $f, g, h \in R[[S, \omega]]$ we have

$$\mathsf{e}_s f = \mathsf{e}_t g \quad \text{and} \quad \mathsf{e}_t(1 - f) = \mathsf{e}_s h. \tag{4.2.5}$$

Since $s \notin tS$, the left cancellativity of S and the first part of (4.2.5) imply that

$$0 = (\mathsf{e}_t g)(s) = (\mathsf{e}_s f)(s) = \omega_s(f(1)),$$

and since ω_s is injective, $f(1) = 0$ follows. Hence, using that $t \notin sS$, from the second part of (4.2.5) we obtain

$$0 = (\mathsf{e}_s h)(t) = (\mathsf{e}_t(1-f))(t) = \omega_t((1-f)(1)) = \omega_t(1 - f(1)) = \omega_t(1) = 1,$$

a contradiction.

(ii) Let $s, t \in S$, and assume that $sS \nsubseteq tS$. Since S is right duo, $st = tp$ for some $p \in S$. Hence $\mathsf{e}_s\mathsf{e}_t = \mathsf{e}_t\mathsf{e}_p$, and since $R[[S,\omega]]$ has weak dimension less than or equal to one, by Lemma 4.2.2 there exist $f, g, h \in R[[S,\omega]]$ such that

$$\mathsf{e}_s f = \mathsf{e}_t g \quad \text{and} \quad (1-f)\mathsf{e}_t = h\mathsf{e}_p. \tag{4.2.6}$$

Since $s \notin tS$ and S is cancellative, it follows from (4.2.6) that $0 = (\mathsf{e}_t g)(s) = (\mathsf{e}_s f)(s) = \omega_s(f(1))$, and since ω_s is injective, we obtain $f(1) = 0$. Hence the second part of (4.2.6) implies that

$$(h\mathsf{e}_p)(t) = [(1-f)\mathsf{e}_t](t) = (1-f)(1) = 1 - f(1) = 1, \tag{4.2.7}$$

and thus $t = xp$ for some $x \in S$. Therefore $tp = st = sxp$, and since S is cancellative, $t = sx$ follows. Hence $tS \subseteq sS$, and it follows that S is a right chain monoid. $\square$

Lemma 4.2.8. *Let $(S, \leqslant)$ be an ordered right chain monoid. Then the order $\leqslant$ is total if and only if for any $s \in S$ we have $s \leqslant 1$ or $s \geqslant 1$.*

Proof. Assume that for any $s \in S$ we have $s \leqslant 1$ or $s \geqslant 1$, and let $x, y \in S$. Since S is a right chain monoid, we may assume that $x = ys$ for some $s \in S$. If $s \leqslant 1$ (resp. $s \geqslant 1$), then $x \leqslant y$ (resp. $x \geqslant y$). Hence the order $\leqslant$ is total. The opposite implication is obvious. $\square$

If $(S, \leqslant)$ is a nontrivial positively ordered monoid, then clearly S is not a group, i.e. $S \setminus U(S) \neq \emptyset$. Therefore, when skew generalized power series rings with positively ordered exponents are considered, as it is in this chapter, then the following lemma gives some necessary conditions for such a power series ring to be right distributive.

Lemma 4.2.9. *Let R be a ring, $(S, \leqslant)$ a strictly ordered left cancellative monoid, and let $\omega : S \to \mathrm{End}(R)$ be a monoid homomorphism such that the ring $R[[S, \omega]]$ is right distributive. Then*

(i) *For any $s \in S \setminus U(S)$ and $a \in R$ there exists $b \in R$ such that $\omega_s(a) = \omega_s(a)ab$.*

(ii) *If ω_{s_0} is injective for some $s_0 \in S \setminus U(S)$, then*

(a) R is strongly regular.

(b) ω_s is bijective for any $s \in S$.

(c) If the order $\leqslant$ is total, then the ring $R[[S, \omega]]$ is reduced.

Proof. (i) Let $s \notin U(S)$ and $a \in R$. By Theorem 2.1.2 there exist $f, g, h, k \in R[[S, \omega]]$ such that $f + g = 1$, $\mathsf{c}_a f = \mathsf{e}_s h$ and $\mathsf{e}_s g = \mathsf{c}_a k$. Note that $st \neq 1$ for any $t \in S$ (otherwise $sts = s$; thus $ts = 1$ by the left cancellativity of S, and we obtain $s \in U(S)$, a contradiction). Hence

$$af(1) = (\mathsf{c}_a f)(1) = (\mathsf{e}_s h)(1) = 0.$$

Since S is left cancellative, we obtain also that

$$\omega_s(g(1)) = (\mathsf{e}_s g)(s) = (\mathsf{c}_a k)(s) = ak(s).$$

Thus

$$\omega_s(a) = \omega_s(a)\omega_s(1) = \omega_s(a)[\omega_s(f(1)) + \omega_s(g(1))] = \omega_s(a)\omega_s(g(1)) = \omega_s(a)ak(s).$$

(ii) (a) This follows from (i) and Lemma 3.7.1.

(b) Let $s \in S$. Then (a), Corollary 4.2.5 and Lemma 3.7.1 imply that ω_s is injective. Thus, to complete the proof it suffices to show that if $a \in R$, then $a \in \omega_s(R)$. By Theorem 2.1.2 there exist $f, h, k \in R[[S,\omega]]$ with $\mathsf{c}_a\mathsf{e}_s f = \mathsf{e}_s h$ and $\mathsf{e}_s(1-f) = \mathsf{c}_a\mathsf{e}_s k$. Therefore, applying also the left cancellativity of S, we obtain

$$a\omega_s(f(1)) = (\mathsf{c}_a\mathsf{e}_s f)(s) = (\mathsf{e}_s h)(s) = \omega_s(h(1)) \in \omega_s(R) \qquad (4.2.8)$$

and

$$1 - \omega_s(f(1)) = (\mathsf{e}_s(1-f))(s) = (\mathsf{c}_a\mathsf{e}_s k)(s) = a\omega_s(k(1)). \qquad (4.2.9)$$

By (a) there exist $u_1, u_2 \in U(R)$ and central idempotents $e_1, e_2 \in R$ such that

$$f(1) = u_1 e_1 \quad \text{and} \quad k(1) = u_2 e_2.$$

From (4.2.8) and Corollary 4.2.5 we obtain $a\omega_s(u_1)e_1 \in \omega_s(R)$, and thus

$$ae_1 \in \omega_s(R)\omega_s(u_1^{-1}) \subseteq \omega_s(R). \qquad (4.2.10)$$

On the other hand, by multiplying (4.2.9) by $1 - e_1$, we obtain

$$1 - e_1 = a\omega_s(u_2)e_2(1 - e_1). \qquad (4.2.11)$$

Now by multiplying (4.2.11) by $1 - e_2$ we obtain $(1 - e_1)(1 - e_2) = 0$, and thus $e_2(1-e_1) = 1-e_1$. Hence by (4.2.11) we have $1-e_1 = a\omega_s(u_2)(1-e_1)$. Therefore

$$a(1 - e_1) = (1 - e_1)\omega_s(u_2^{-1}) = \omega_s((1 - e_1)u_2^{-1}) \in \omega_s(R),$$

which together with (4.2.10) implies that $a \in \omega_s(R)$.

(c) By Theorem 8.3.12, to prove that $R[[S,\omega]]$ is reduced, it suffices to show that for any $s \in S$ and $a \in R$, $a\omega_s(a) = 0$ implies $a = 0$. But this is obvious, since by (a), a is a product of a unit and a central idempotent e, and by Corollary 4.2.5 we have $\omega_s(e) = e$. □

In the next lemma we give some necessary conditions for a skew generalized power series ring to be right duo.

Lemma 4.2.10. *Let R be a ring, $(S,\leqslant)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that the ring $R[[S,\omega]]$ is right duo. Then*

(i) The ring R and the monoid S are right duo, and ω_s is idempotent-stabilizing for any $s \in S$.

(ii) If $s \in S$ is left cancellative, then ω_s is bijective.

Proof. (i) Since $A = R[[S,\omega]]$ is right duo, for any $a, b \in R$ there exists $f \in A$ such that $\mathsf{c}_{ba} = \mathsf{c}_b\mathsf{c}_a = \mathsf{c}_a f$. Hence $ba = \mathsf{c}_{ba}(1) = (\mathsf{c}_a f)(1) = \mathsf{c}_a(1)f(1) = af(1) \in aR$ proves that R is right duo. Similarly, for any $s, t \in S$ there exists $g \in A$ with $\mathsf{e}_{st} = \mathsf{e}_s\mathsf{e}_t = \mathsf{e}_t g$. Now $(\mathsf{e}_t g)(st) = \mathsf{e}_{st}(st) = 1$ implies that $st \in tS$, and thus S is right duo. Since any right duo ring is abelian, Lemma 4.2.4 completes the proof of (i).

(ii) Set $A = R[[S,\omega]]$ and assume that $s \in S$ is left cancellative. We first show that ω_s is surjective. Since A is right duo, for any $r \in R$ there exists $h \in A$ such that $\mathsf{c}_r\mathsf{e}_s = \mathsf{e}_s h$. Hence, using also that s is left cancellative, we obtain $r = (\mathsf{c}_r\mathsf{e}_s)(s) = (\mathsf{e}_s h)(s) = \omega_s(h(1)) \in \omega_s(R)$, and thus ω_s is a surjection.

To prove that ω_s is injective, we adapt the proof of [51, Theorem 1]. The case where $s \in U(S)$ is obvious. Thus we assume that $s \notin U(S)$. Let $a \in R$ be such that $\omega_s(a) = 0$. Since ω_s is surjective, $a = \omega_s(b)$ for some $b \in R$. Since A is right duo, there exists $k \in A$ such that

$$\mathsf{c}_a\mathsf{e}_s + \mathsf{e}_{s^3} = \mathsf{e}_s(\mathsf{c}_b + \mathsf{c}_a\mathsf{e}_s + \mathsf{e}_{s^2}) = (\mathsf{c}_b + \mathsf{c}_a\mathsf{e}_s + \mathsf{e}_{s^2})k. \tag{4.2.12}$$

Since s is left cancellative and $s \notin U(S)$, taking values of (4.2.12) at 1, s and s^3, respectively, we obtain the following equations:

$$0 = bk(1), \quad a = bk(s) + a\omega_s(k(1)), \quad 1 = bk(s^3) + a\omega_s(k(s^2)) + \omega_{s^2}(k(s)). \tag{4.2.13}$$

From the first part of (4.2.13) we obtain $0 = \omega_s(bk(1)) = a\omega_s(k(1))$, and thus the second part of (4.2.13) implies that $a = bk(s)$, which leads to $0 = a\omega_s(k(s))$. Applying ω_{s^2} to the third equation of (4.2.13), we obtain $\omega_{s^4}(k(s)) = 1$. Hence, since A is right duo, it follows that $\mathsf{e}_{s^3} = \mathsf{e}_{s^3}\mathsf{c}_{\omega_s(k(s))} \in \mathsf{c}_{\omega_s(k(s))}A$, and thus $1 = \omega_s(k(s))d$ for some $d \in R$. Hence $a = a\omega_s(k(s))d = 0 \cdot d = 0$, proving that ω_s is bijective. $\square$

In the case when the coefficient ring R is a finite direct product of rings, the following result will allow us to represent the ring $R[[S,\omega]]$ as a direct product of skew generalized power series rings.

Proposition 4.2.11. *Let $R_1, R_2, \ldots, R_n$ be rings and let $R = \prod_{i=1}^n R_i$. For any $i \in \{1, 2, \ldots, n\}$ let $\tau_i : R_i \to R$ and $\pi_i : R \to R_i$ be the natural injection and the natural projection, respectively. Let $(S, \leqslant)$ be a strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that $\omega_s \circ \tau_i(R_i) \subseteq \tau_i(R_i)$ for any $s \in S$ and $i \in \{1, 2, \ldots, n\}$. Then for every $i \in \{1, 2, \ldots, n\}$ the map $\omega_i : S \to \mathrm{End}(R_i)$ defined by*

$$\omega_{i,s} = \pi_i \circ \omega_s \circ \tau_i \qquad \textit{for any } s \in S$$

is a monoid homomorphism and the ring $R[[S,\omega]]$ is isomorphic to the ring $\prod_{i=1}^n R_i[[S,\omega_i]]$.

Proof. Since by assumption for any $i \in \{1, 2, \ldots, n\}$ and $s \in S$ we have $\omega_s \circ \tau_i(R_i) \subseteq \tau_i(R_i)$, it easily follows that $\omega_{i,s}(1) = 1$. Now to complete the proof, it suffices to repeat arguments of the proof of [57, Proposition 2.1]. $\square$

In the proof of Theorem 4.2.15 we will need the following characterization of finite products of division rings (see [55, Corollary 13]. Recall that a ring R is said to be *orthogonally finite* if R has no infinite set of mutually orthogonal idempotents.

Lemma 4.2.12. *A ring R is orthogonally finite strongly regular if and only if R is a finite direct product of division rings.*

Lemma 4.2.13. *Let $(S, \leqslant)$ be a right chain positively strictly ordered monoid, and let $t \in S$. Then*

(i) For any $s \in S$ there exists a unique element $s^{(t)} \in S$ such that $st = ts^{(t)}$.

(ii) Let R be a ring, and $\omega : S \to \operatorname{End}(R)$ a monoid homomorphism such that ω_t is bijective. For any $f \in R[[S, \omega]]$ define $f^{(t)} : S \to R$ by $f^{(t)}(x) = \omega_t^{-1}(f(s))$ if $x = s^{(t)}$ for some $s \in S$, and $f^{(t)}(x) = 0$ otherwise. Then $f^{(t)} \in R[[S, \omega]]$ and $f \mathsf{e}_t = \mathsf{e}_t f^{(t)}$.

Proof. We claim that for any $s \in S$, $st \in tS$. Otherwise, since S is a right chain monoid, $t = stv$ for some $v \in S \setminus \{1\}$. Since $(S, \leqslant)$ is positively ordered, $s \geqslant 1$ and $v > 1$, and we obtain $t = stv > t$, a contradiction that proves our claim. Hence there exists $s^{(t)} \in S$ such that $st = ts^{(t)}$. Since by Lemma 4.2.8 the order $\leqslant$ is total, such an element $s^{(t)}$ is unique, and the proof of (i) is complete. Furthermore, for any $s_1, s_2 \in S$ we have $s_1 \leqslant s_2 \Leftrightarrow s_1^{(t)} \leqslant s_2^{(t)}$. Thus for any $f \in R[[S, \omega]]$ the map $f^{(t)} : S \to R$ is well-defined and $f^{(t)} \in R[[S, \omega]]$. The rest of (ii) is easy to verify. □

Recall that a monoid S is *cyclic* if for some $s \in S$ we have $S = \{s^n : n \in \mathbb{N} \cup \{0\}\}$. In our investigations in present chapter we will use the following (see [55, Lemma 7]):

Lemma 4.2.14. *Let S be a positively strictly ordered monoid which is right chain. Then S is not cyclic if and only if S contains an infinite sequence of elements $t, s_1, s_2, s_3, \ldots$ such that*

$$s_1 < s_2 < s_3 < \ldots < t. \tag{4.2.14}$$

Proof. By Lemma 4.2.8, S is linearly ordered. Assume that S is not cyclic. We start with the case when the set $S \setminus \{1\}$ contains a minimal element s. Since S is not cyclic, there exists $t \in S \setminus \{1, s, s^2, s^3, \ldots\}$. If the sequence $s < s^2 < s^3 < \ldots$ is not bounded by t, then for some $i \geqslant 1$ we have $s^i < t < s^{i+1}$ and thus for some $x, y \in S \setminus \{1\}$, $t = s^i x$ and $s^{i+1} = ty$. Hence $s^{i+1} = s^i xy$, which leads to

$s = xy \geqslant s^2$, a contradiction. Therefore, for every i, $s^i < t$, and putting $s_i = s^i$ we get a sequence (4.2.14). We are left with the case when $S \setminus \{1\}$ contains no minimal element. Then starting with any $t \in S \setminus \{1\}$ we find in S a sequence $1 < \ldots < a_3 < a_2 < a_1 < t$. Since $a_i < t$, for any i there exists s_i with $t = a_i s_i$ and $s_i < s_{i+1} < t$ easily follows, proving the existence of a sequence (4.2.14).

Clearly, if S contains a sequence (4.2.14), then S is not cyclic. □

Theorem 4.2.15. *Let R be a ring, $(S, \leqslant)$ a nontrivial positively strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Then the following conditions are equivalent:*

(1) $R[[S,\omega]]$ is right duo right distributive.

(2) $R[[S,\omega]]$ is reduced right distributive.

(3) $R[[S,\omega]]$ is right distributive and ω_s is injective for any $s \in S$.

(4) $R[[S,\omega]]$ is right duo right Bézout.

(5) $R[[S,\omega]]$ is reduced right Bézout.

(6) $R[[S,\omega]]$ is right quasi-duo right Bézout and ω_s is injective for any $s \in S$.

(7) $R[[S,\omega]]$ is semicommutative right Bézout and ω_s is injective for any $s \in S$.

(8) $R[[S,\omega]]$ has weak dimension less than or equal to one and is right duo.

(9) $R[[S,\omega]]$ has weak dimension less than or equal to one, R is abelian, S is a right chain monoid, and ω_s is bijective and idempotent-stabilizing for any $s \in S$.

(10) ω_s is bijective and idempotent-stabilizing for any $s \in S$, and either

(a) *S is cyclic and R is countable-injective strongly regular*

or

(b) *S is not cyclic, S is a right chain monoid and R is a finite direct product of division rings.*

Proof. Set $A = R[[S, \omega]]$.

(8) $\Rightarrow$ (9) Since S is positively strictly ordered, S is cancellative by Lemma 4.2.3. Thus Lemmas 4.2.10 and 4.2.7(ii) imply that ω_s is bijective and idempotent-stabilizing for any $s \in S$, and that S is a right chain monoid. Moreover, R is right duo by Lemma 4.2.10(i), and thus R is abelian.

(9) $\Rightarrow$ (10) If S is cyclic, say generated by s, then $A \cong R[[x; \sigma]]$, where $\sigma = \omega_s$, and in this case this implication follows from Theorem 3.6.1. Therefore, we assume that S is not cyclic, and we show that if (9) holds, then condition (b) of (10) is satisfied.

To prove (b), we will apply Lemma 4.2.12. We first show that the ring R is strongly regular. Let $a \in R$ and choose any $s \in S \setminus \{1\}$. Since $\mathsf{c}_a \mathsf{e}_s = \mathsf{e}_s \mathsf{c}_{\omega_s^{-1}(a)}$, Lemma 4.2.2 implies that $\mathsf{c}_a f = \mathsf{e}_s g$ and $(1-f)\mathsf{e}_s = h\mathsf{c}_{\omega_s^{-1}(a)}$ for some $f, g, h \in A$. Hence

$$af(1) = (\mathsf{c}_a f)(1) = (\mathsf{e}_s g)(1) = \mathsf{e}_s(1)g(1) = 0,$$

and since S is cancellative by Lemma 4.2.3, we obtain also that

$$1 - f(1) = [(1-f)\mathsf{e}_s](s) = (h\mathsf{c}_{\omega_s^{-1}(a)})(s) = h(s)a.$$

Thus $a = a(1 - f(1)) = ah(s)a$, which proves that R is von Neumann regular. Since R is abelian, it follows that R is strongly regular.

Now we show that R is a finite direct product of division rings. By Lemma 4.2.12, we need only prove that R is orthogonally finite. Suppose, for a contradiction, that there exists an infinite sequence $e_1, e_2, e_3, \ldots$ of nonzero orthogonal idempotents of R. By Lemma 4.2.14, in S there exist an element t and a sequence $(s_n)_{n\in\mathbb{N}}$ such that

$$s_1 < s_2 < s_3 < \ldots < t.$$

Define $p \in A$ by $p(s_i) = e_i$ for all $i \in \mathbb{N}$, and $p(x) = 0$ for $x \in S \setminus \{s_1, s_2, s_3, \cdots\}$, and let $p^{(t)} \in A$ be defined as in Lemma 4.2.13(ii). Then by Lemma 4.2.13 we have $p\mathsf{e}_t = \mathsf{e}_t p^{(t)}$, and by Lemma 4.2.2 there exist $f, g, h \in A$ with

$$pf = \mathsf{e}_t g \quad \text{and} \quad \mathsf{e}_t = f\mathsf{e}_t + hp^{(t)}. \tag{4.2.15}$$

We claim that

$$e_j f(1) = 0 \quad \text{for any} \quad j \in \mathbb{N}. \tag{4.2.16}$$

To see this, note that since S is positively ordered and $s_j < t$, we have $s_j \notin tS$, and thus from the first part of (4.2.15) we obtain

$$0 = (\mathsf{e}_t g)(s_j) = (pf)(s_j) = p(s_j)\omega_{s_j}(f(1)) + \sum_{\substack{(x,y)\in \mathrm{X}_{s_j}(p,f) \\ x \neq s_j}} p(x)\omega_x(f(y)) =$$

$$= e_j\omega_{s_j}(f(1)) + e_{k_1}\omega_{s_{k_1}}(f(y_1)) + e_{k_2}\omega_{s_{k_2}}(f(y_2)) + \cdots + e_{k_m}\omega_{s_{k_m}}(f(y_m))$$

for some $m \in \mathbb{N}$, $y_1, y_2, \ldots, y_m \in S$, and $k_1, k_2, \ldots, k_m \in \mathbb{N} \setminus \{j\}$. Multiplying the above equation by e_j from the left, we obtain

$$0 = e_j\omega_{s_j}(f(1)) = \omega_{s_j}(e_j)\omega_{s_j}(f(1)) = \omega_{s_j}(e_j f(1)).$$

Since ω_{s_j} is injective, $e_j f(1) = 0$ follows, completing the proof of (4.2.16).

On the other hand, applying the definition of $p^{(t)}$ and the second part of (4.2.15), we obtain

$$1 = \mathsf{e}_t(t) = (f\mathsf{e}_t)(t) + (hp^{(t)})(t) = f(1) + \sum_{(x,y)\in \mathrm{X}_t(h,p^{(t)})} h(x)\omega_x(p^{(t)}(y)) =$$

$$= f(1) + h(x_1)e_{i_1} + h(x_2)e_{i_2} + \cdots + h(x_n)e_{i_n}$$

for some $n, i_1, \ldots, i_n \in \mathbb{N} \cup \{0\}$ and $x_1, x_2, \ldots, x_n \in S$. Take any $j \in \mathbb{N} \setminus \{i_1, i_2, \ldots, i_n\}$. Since $e_{i_d} e_j = 0$ for all $d \in \{1, 2, \ldots, n\}$, from the above equation it follows that $e_j = f(1)e_j$. But $e_j f(1) = 0$ by (4.2.16), and we obtain $e_j = e_j f(1) e_j = 0$, a contradiction.

(10) $\Rightarrow$ (1) If S is cyclic, then this implication follows from Theorem 3.6.1. As-

sume that S is not cyclic. Then $R = D_1 \times \cdots \times D_n$ for some division rings $D_1, \ldots, D_n$. Furthermore, if $s \in S$, then $\omega_s(e) = e$ for any idempotent $e \in R$, and thus $\omega_s \circ \tau_i(D_i) \subseteq \tau_i(D_i)$ for any $1 \leqslant i \leqslant n$, where $\tau_i : D_i \to R$ is the natural injection. Hence by Proposition 4.2.11 we have $A \cong D_1[[S, \omega_1]] \times \cdots \times D_n[[S, \omega_n]]$, where for any $i \in \{1, \ldots, n\}$, $\omega_i : S \to \mathrm{End}(D_i)$ is a monoid homomorphism such that $\omega_{i,s} = \omega_i(s)$ is bijective for any $s \in S$. Since S is a right chain monoid, the order $\leqslant$ is total by Lemma 4.2.8, and thus [56, Theorem 4.7] implies that for any $i \in \{1, \ldots, n\}$, $D_i[[S, \omega_i]]$ is a right chain ring and any nonzero principal right ideal of this ring is generated by e_t for some $t \in S$. Thus by Lemma 4.2.13(ii), $D_i[[S, \omega_i]]$ is a right duo ring for any $i \in \{1, \ldots, n\}$. Therefore, being a finite direct product of right chain right duo rings, A is a right distributive right duo ring.

(1) $\Rightarrow$ (4) Since S is left cancellative by Lemma 4.2.6, it follows from Lemmas 4.2.10(ii) and 4.2.9(ii) that R is strongly regular and $J(R) = 0$ follows. Hence Lemma 4.2.1 implies that $A/J(A)$ is strongly regular, and thus R is right Bézout by Proposition 3.1.3(ii).

(4) $\Rightarrow$ (6) Proposition 3.1.3(ii) implies that A is right distributive, and thus S is left cancellative by Lemma 4.2.6. Now (6) follows from Lemma 4.2.10(ii).

(6) $\Rightarrow$ (5) Proposition 3.1.3(ii) implies that A is right distributive. From Lemmas 4.2.6, 4.2.7(i) and 4.2.8 it follows that the order $\leqslant$ is total, and thus by Lemma 4.2.9(ii)(c), A is reduced.

(5) $\Rightarrow$ (7) We show first that for any $s \in S$, ω_s is injective. For this, assume that $a \in R$ and $\omega_s(a) = 0$. Then in A we have $(\mathsf{c}_a \mathsf{e}_s)^2 = \mathsf{c}_{a\omega_s(a)} \mathsf{e}_{s^2} = 0$, and since A is reduced, $\mathsf{c}_a \mathsf{e}_s = 0$ follows. Hence $a = (\mathsf{c}_a \mathsf{e}_s)(s) = 0$, which proves that ω_s is injective. Since every reduced ring is semicommutative, so is A.

(7) $\Rightarrow$ (3) By Proposition 3.1.3(i) and Lemma 4.2.1, it suffices to show that R is strongly regular. For this, consider any $a \in R$. Since S is nontrivial, there exists $s \in S \setminus \{1\}$. Since A is right Bézout, there exist $f, g, h, k \in A$ with

$\mathsf{c}_a = (\mathsf{c}_a f + \mathsf{e}_s g)h$ and $\mathsf{e}_s = (\mathsf{c}_a f + \mathsf{e}_s g)k$. Since S is positively ordered, it follows that

$$a = \mathsf{c}_a(1) = [(\mathsf{c}_a f + \mathsf{e}_s g)h](1) = (\mathsf{c}_a fh)(1) + (\mathsf{e}_s gh)(1) = af(1)h(1),$$

and thus

$$\omega_s(a) = \omega_s(a)\omega_s(f(1))\omega_s(h(1)). \tag{4.2.17}$$

Moreover

$$0 = \mathsf{e}_s(1) = [(\mathsf{c}_a f + \mathsf{e}_s g)k](1) = (\mathsf{c}_a fk)(1) + (\mathsf{e}_s gk)(1) = af(1)k(1),$$

and thus

$$0 = \omega_s(a)\omega_s(f(1))\omega_s(k(1)). \tag{4.2.18}$$

Furthermore,

$$1 = \mathsf{e}_s(s) = [(\mathsf{c}_a f + \mathsf{e}_s g)k](s) = (\mathsf{c}_a fk)(s) + (\mathsf{e}_s gk)(s) = a(fk)(s) + \omega_s(g(1))\omega_s(k(1)).$$

By (4.2.18), $\omega_s(k(1))$ belongs to the right annihilator of $\omega_s(a)\omega_s(f(1))$, which by assumption is an ideal of R, and thus using (4.2.17) we obtain

$$\omega_s(a)\omega_s(g(1))\omega_s(k(1)) = [\omega_s(a)\omega_s(f(1))\omega_s(h(1))]\omega_s(g(1))\omega_s(k(1)) =$$

$$= \omega_s(a)\omega_s(f(1))[\omega_s(h(1))\omega_s(g(1))]\omega_s(k(1)) = 0.$$

Hence

$$\omega_s(a) = \omega_s(a)1 = \omega_s(a)[a(fk)(s) + \omega_s(g(1))\omega_s(k(1))] = \omega_s(a)a(fk)(s).$$

Thus by Lemma 3.7.1 the ring R is strongly regular.

(3) $\Rightarrow$ (2) follows from Lemmas 4.2.6, 4.2.7(i), 4.2.8 and 4.2.9(ii)(c).

(2) $\Rightarrow$ (10) The same argument as in the proof of (5) $\Rightarrow$ (7) implies that all the ω_s's are injective. Since S is left cancellative by Lemma 4.2.6, it follows

from Lemma 4.2.9(ii) and Corollary 4.2.5 that for any $s \in S$, ω_s is bijective and idempotent-stabilizing.

If S is cyclic, then part (a) of (10) follows from Theorem 3.6.1. Thus we assume that S is not cyclic and prove part (b). By Lemma 4.2.7(i), S is a right chain monoid. By Lemmas 4.2.9(ii)(a) and 4.2.12, to prove the rest of (b) it suffices to show that R is orthogonally finite. For this it suffices to modify slightly the proof of the implication (9) $\Rightarrow$ (10). Suppose, for a contradiction, that there exists an infinite sequence $e_1, e_2, e_3, \ldots$ of nonzero mutually orthogonal idempotents of R. Since S is a positively ordered right chain monoid that is not cyclic, by Lemma 4.2.14 in S there exist an element t and a sequence $(s_n)_{n \in \mathbb{N}}$ such that

$$s_1 < s_2 < s_3 < \ldots < t.$$

Define $p \in A$ by $p(s_i) = e_i$ for all $i \in \mathbb{N}$, and $p(x) = 0$ for $x \in S \setminus \{s_1, s_2, s_3, \ldots\}$. Since A is right distributive, by Theorem 2.1.2 there exist $f, g, h \in A$ with

$$pf = \mathsf{e}_t g \quad \text{and} \quad \mathsf{e}_t = \mathsf{e}_t f + ph. \tag{4.2.19}$$

From the second part of (4.2.19) we obtain

$$1 = \mathsf{e}_t(t) = (\mathsf{e}_t f)(t) + (ph)(t) = \omega_t(f(1)) + \sum_{(x,y) \in \mathrm{X}_t(p,h)} p(x)\omega_x(h(y)) =$$

$$= \omega_t(f(1)) + p(x_1)\omega_{x_1}(h(y_1)) + p(x_2)\omega_{x_2}(h(y_2)) + \cdots + p(x_n)\omega_{x_n}(h(y_n)) =$$

$$= \omega_t(f(1)) + e_{i_1}\omega_{x_1}(h(y_1)) + e_{i_2}\omega_{x_2}(h(y_2)) + \cdots + e_{i_n}\omega_{x_n}(h(y_n))$$

for some $n, i_1, \ldots, i_n \in \mathbb{N} \cup \{0\}$ and $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in S$. Take any $j \in \mathbb{N} \setminus \{i_1, i_2, \ldots, i_n\}$. Since $e_j e_{i_d} = 0$ for all $d \in \{1, 2, \ldots, n\}$, from the above equation it follows that $e_j = e_j\omega_t(f(1)) = \omega_t(e_j f(1))$. But the first part of (4.2.19) implies that $e_j f(1) = 0$ (see the proof of (4.2.16)), and we obtain $e_j = \omega_t(0) = 0$, a contradiction.

(1) $\Rightarrow$ (8) We already know that (1) implies (5), and thus to get (8) it suffices to

apply [80, 4.21(2)]. □

Now we want to point out that the "positively ordered" assumption is essential in Theorem 4.2.15, i.e. if $(S, \leqslant)$ is not assumed to be positively ordered, then conditions (1)–(10) in Theorem 4.2.15 need not be equivalent. For instance, if R is a commutative artinian chain ring that is not reduced, and $(S, \leqslant)$ is a nontrivial totally ordered commutative group, and $\omega : S \to \mathrm{End}(R)$ is the trivial monoid homomorphism, then [56, Theorem 4.6] implies that $R[[S, \omega]]$ is a commutative chain ring that is not reduced, and thus any of the conditions (1), (3), (4), (6), (7) is satisfied but none of the conditions (2), (5), (10) holds. For a more concrete example, one can consider $R = \mathbb{Z}/4\mathbb{Z}$, the ring of integers modulo 4, and $S = \mathbb{Z}$, the additive group of integers with its natural total order $\leqslant$.

Example 3.6.7 shows that we can not add the condition "$R[[x; \sigma]]$ is right semihereditary and right duo" in formulation of Theorem 3.6.1. The diffrent situation is when we consider generalized power series rings $R[[S, \omega]]$ for $(S, \leqslant)$ being a not cyclic nontrivial positively strictly ordered monoid. Namely we have the following:

Proposition 4.2.16. *Let R is a ring, $(S, \leqslant)$ a not cyclic positively strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Then the following conditions are equivalent:*

(1) $R[[S, \omega]]$ is a right semihereditary and right duo ring.

(2) S is a right chain monoid, R is a finite direct product of division rings and ω_s is bijective and idempotent-stabilizing for any $s \in S$.

Proof. (1) $\Rightarrow$ (2) It is obvious by Theorems 3.6.6 and 4.2.15

(2) $\Rightarrow$ (1) By Theorem 4.2.15 the ring $R[[S, \omega]]$ has weak dimension less than or equal to one and is right duo. Thus Theorem 3.6.6 implies that to prove (1) it is enough to show that $R[[S, \omega]]$ is right Rickartian.

To show that $R[[S, \omega]]$ is right Rickartian we will use Remark 3.6.5. Using the same arguments as in the proof of implication (10) $\Rightarrow$ (3) of Theorem 4.2.15, we can show that (2) implies that $R[[S,\omega]] \cong D_1[[S,\omega_1]] \times \cdots \times D_n[[S,\omega_n]]$, where for any $i \in \{1,\dots,n\}$, $\omega_i : S \to \mathrm{End}(D_i)$ is a monoid homomorphism such that $\omega_{i,s} = \omega_i(s)$ is bijective for any $s \in S$ and for any $i \in \{1,\dots,n\}$, $D_i[[S,\omega_i]]$ is a right chain ring and D_i is division ring. Since by Lemma 4.2.8 the monoid S is totally ordered and every ω_i is bijective, it is easy to see that $D_i[[S,\omega_i]]$ is domain for every i. So we deduce that for every $f \in R[[S,\omega]]$, the right annihilator of f is generated by an idempotent and the fact that $R[[S,\omega]]$ is right Rickartian follows. □

4.3 Right Gaussian skew generalized power series rings

In this section we will extend Theorem 3.7.2 to skew generalized power series rings with exponents in a positively ordered right chain monoid.

Theorem 4.3.1. *Let R be a ring, $(S,\leqslant)$ a nontrivial positively strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Then the following conditions are equivalent:*

(1) $R[[S,\omega]]$ is a right Gaussian ring and S is a right chain monoid.

(2) $R[[S,\omega]]$ is right duo right distributive.

(3) $R[[S,\omega]]$ is reduced right distributive.

(4) $R[[S,\omega]]$ is right distributive and ω_s is injective for any $s \in S$.

(5) $R[[S,\omega]]$ has weak dimension less than or equal to one and is right duo.

(6) $R[[S,\omega]]$ is right duo right Bézout.

(7) $R[[S,\omega]]$ is reduced right Bézout.

(8) $R[[S,\omega]]$ is right quasi-duo right Bézout and ω_s is injective for any $s \in S$.

(9) $R[[S,\omega]]$ is semicommutative right Bézout and ω_s is injective for any $s \in S$.

(10) For any $s \in S$, ω_s is bijective and idempotent-stabilizing, and either

(a) *S is cyclic and R is countable-injective strongly regular*

or

(b) *S is not cyclic, S is a right chain monoid and R is a finite direct product of division rings.*

Since the equivalence of conditions (2) through (10) of the above theorem has been already established in Theorem 4.2.15, to prove the result it suffices to prove the implications (1) $\Rightarrow$ (10) and (2) $\Rightarrow$ (1). This will be done with the aid of the following two lemmas which give some necessary conditions for the skew generalized power series ring $R[[S,\omega]]$ to be right Gaussian.

Lemma 4.3.2. *Let R be a ring, $(S,\leqslant)$ a nontrivial positively strictly ordered right chain monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that the ring $R[[S,\omega]]$ is right Gaussian. Then*

(i) For any $s \in S$, ω_s is bijective and idempotent-stabilizing.

(ii) R is strongly regular.

Proof. (i) Since $R[[S,\omega]]$ is right Gaussian, $R[[S,\omega]]$ is right duo by Lemma 2.2.5. Thus Lemma 4.2.10(i) implies that ω_s is idempotent-stabilizing for any $s \in S$. Furthermore, since S is a positively strictly ordered right chain monoid, it follows from Lemma 4.2.10(ii) that all the ω_s's are bijective.

(ii) Since S is nontrivial, there exists $s \in S \setminus \{1\}$. By (i), ω_s is injective, and thus by Lemma 3.7.1, to prove that R is strongly regular, it suffices to show that for any $a \in R$ we have $\omega_s(a) \in \omega_s(a)aR$. For that, set $A = R[[S,\omega]]$. Since A is right Gaussian and in $A[x]$ we have

$$(\mathsf{c}_{\omega_s(a)} + \mathsf{e}_s x)(\mathsf{c}_a - \mathsf{e}_s x) = \mathsf{c}_{\omega_s(a)a} - \mathsf{e}_{s^2} x^2,$$

it follows that

$$\mathsf{c}_{\omega_s(a)}\mathsf{e}_s = \mathsf{c}_{\omega_s(a)a} f + \mathsf{e}_{s^2} g \quad \text{for some} \quad f, g \in A. \tag{4.3.1}$$

Since S is positively strictly ordered right chain monoid and $s \neq 1$, we deduce that $s \notin s^2S$. Hence from (4.3.1) we obtain

$$\omega_s(a) = (\mathsf{c}_{\omega_s(a)}\mathsf{e}_s)(s) = (\mathsf{c}_{\omega_s(a)a}f)(s) + (\mathsf{e}_{s^2}g)(s) = \omega_s(a)af(s) \in \omega_s(a)aR,$$

as desired. $\square$

Lemma 4.3.3. *Let R be a ring, $(S, \leqslant)$ a positively strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that the ring $R[[S,\omega]]$ is right Gaussian. If S contains an infinite sequence of elements $t, s_1, s_2, s_3, \ldots$ such that*

$$s_1 < s_2 < s_3 < \ldots < t,$$

then R is a finite direct product of division rings.

Proof. By Lemma 4.3.2(i), R is strongly regular, and thus by Lemma 4.2.12, to prove the result it suffices to show that R contains no infinite sequence of nonzero mutually orthogonal idempotents. Suppose, for a contradiction, that there exists an infinite sequence $e_1, e_2, e_3, \ldots$ of nonzero mutually orthogonal idempotents of R. By assumption, in S there exist an element t and a sequence $(s_n)_{n\in\mathbb{N}}$ such that

$$s_1 < s_2 < s_3 < \ldots < t.$$

Set $A = R[[S,\omega]]$, and define $p \in A$ by $p(s_i) = e_i$ for all $i \in \mathbb{N}$, and $p(x) = 0$ for $x \in S \setminus \{s_1, s_2, s_3, \ldots\}$. Since A is right Gaussian, A is right duo by Lemma 2.2.5, and thus there exists $h \in A$ such that $p\mathsf{e}_t = \mathsf{e}_t h$. We claim that

there exists an infinite sequence $u_1, u_2, u_3, \ldots$ of different elements of $\mathrm{supp}(h)$

such that $s_i t = tu_i$ for any $i \in \mathbb{N}$.

To prove the claim, note first that if $i, j \in \mathbb{N}$ and $i \neq j$, then $s_i t \neq s_j t$. Indeed, if $i < j$, then $s_i < s_j$, and $s_i t < s_j t$. Similarly we obtain $s_i t > s_j t$ in the case where

$i > j$. Applying this observation, we obtain

$$0 \neq e_i = p(s_i)\omega_{s_i}(\mathsf{e}_t(t)) = (p\mathsf{e}_t)(s_it) = (\mathsf{e}_th)(s_it).$$

Hence there exists $u_i \in \operatorname{supp}(h)$ with $s_it = tu_i$, and furthermore $u_i = u_j$ if and only if $i = j$.

Now we define a function $k : S \to R$ as follows: $k(u_i) = e_i$ for any $i \in \mathbb{N}$, and $k(x) = 0$ for any $x \in S \setminus \{u_1, u_2, u_3, \ldots\}$. Since $\operatorname{supp}(k) \subseteq \operatorname{supp}(h)$, it follows that $k \in A$. Furthermore, using that ω_t is idempotent-stabilizing by Lemma 4.3.2(i), it is easy to see that $p\mathsf{e}_t = \mathsf{e}_tk$. Since in the ring $A[x]$ we have

$$(p + \mathsf{e}_tx)(-\mathsf{e}_t + kx) = -p\mathsf{e}_t + (pk - \mathsf{e}_{t^2})x + p\mathsf{e}_tx^2$$

and A is right Gaussian, there exist $f, g \in A$ with

$$\mathsf{e}_{t^2} = p\mathsf{e}_tf + pkg - \mathsf{e}_{t^2}g. \tag{4.3.2}$$

Since $\mathsf{e}_{t^2} = p(\mathsf{e}_tf + kg) - \mathsf{e}_{t^2}g$ and S is positively strictly ordered, it follows that

$$1 = \mathsf{e}_{t^2}(t^2) = [p(\mathsf{e}_tf + kg)](t^2) - (\mathsf{e}_{t^2}g)(t^2) = e_{i_1}d_1 + \cdots + e_{i_n}d_n - \omega_{t^2}(g(1)) \tag{4.3.3}$$

for some $n, i_1, \ldots, i_n \in \mathbb{N}$ and $d_1, \ldots, d_n \in R$. Take any $j \in \mathbb{N} \setminus \{i_1, i_2, \ldots, i_n\}$. Since $e_je_{i_q} = 0$ for all $q \in \{1, 2, \ldots, n\}$, by multiplying (4.3.3) by e_j from the left, we obtain $e_j = -e_j\omega_{t^2}(g(1))$. Since furthermore ω_{t^2} is idempotent-stabilizing and bijective by Lemma 4.3.2(i), we deduce that $e_j = -e_jg(1)$.

Next we take values of the summands of (4.3.2) on s_ju_j, and multiply them by e_j from the left. Since $s_j < t$, we have $s_ju_j < tu_j = s_jt < t^2$. Thus $\mathsf{e}_{t^2}(s_ju_j) = 0$, and consequently

$$e_j \cdot \mathsf{e}_{t^2}(s_ju_j) = 0. \tag{4.3.4}$$

It is clear that if $(p\mathsf{e}_tf)(s_ju_j)$ is non-zero, then it is a sum of elements of the form $p(s_i) \cdot \omega_{s_i}(\mathsf{e}_t(t)) \cdot \omega_{s_it}(f(v))$, where $s_itv = s_ju_j$. If we would have $s_i = s_j$, then since S is positively ordered, $s_it \leqslant s_itv = s_iu_i < tu_i = s_it$, a contradiction. Thus $s_i \neq s_j$, hence $p(s_i) = e_i \neq e_j$, and $e_jp(s_i) = 0$ follows, which shows that

$$e_j \cdot (p\mathsf{e}_t f)(s_ju_j) = 0. \tag{4.3.5}$$

Now we consider the value $e_j \cdot (pkg)(s_ju_j)$. It is clear that $(pkg)(s_ju_j)$ is a sum of elements of the form $z = p(s_i) \cdot \omega_{s_i}(k(u_l)) \cdot \omega_{s_iu_l}(g(v))$, where $s_iu_lv = s_ju_j$. If $i \neq j$, then $p(s_i) = e_i \neq e_j$, and $e_jz = 0$ follows. Similarly, if $l \neq j$, then $k(u_l) = e_l \neq e_j$, and again $e_jz = 0$. We are left with the case where $i = l = j$. Then $s_ju_jv = s_ju_j$, and since S is positively ordered, $v = 1$ follows. Hence $e_jz = e_j \cdot \omega_{s_ju_j}(g(1)) = \omega_{s_ju_j}(e_jg(1))$. We already know that $e_jg(1) = -e_j$, and thus

$$e_j \cdot (pkg)(s_ju_j) = -e_j. \tag{4.3.6}$$

Finally, we consider the value $e_j \cdot (\mathsf{e}_{t^2}g)(s_ju_j)$. Since $s_ju_j < t^2$ and S is positively ordered, it follows that $s_ju_j \notin t^2S$, and thus $(\mathsf{e}_{t^2}g)(s_ju_j) = 0$. Hence

$$e_j \cdot (\mathsf{e}_{t^2}g)(s_ju_j) = 0. \tag{4.3.7}$$

Now from equations (4.3.2) and (4.3.4) – (4.3.7) we obtain $e_j = 0$, a contradiction.

□

We are now ready to prove Theorem 4.3.1.

Proof of Theorem 4.3.1 As we have already noted, conditions (2)–(10) are equivalent by Theorem 4.2.15. In particular, by the equivalence (2) $\Leftrightarrow$ (10), (2) implies that S is a right chain monoid and thus the implication (2) $\Rightarrow$ (1) follows from Theorem 2.2.11. Hence, to complete the proof it suffices to show that (1) implies (10).

Assume (1). Then by Lemma 4.3.2(i), ω_s is bijective and idempotent-stabilizing for any $s \in S$. To show that (a) or (b) of (10) holds, we consider two cases, depending on whether S is cyclic or not. In the first case assume that S is generated by an element $s \in S$, and set $\sigma = \omega_s$. Since S is nontrivial and strictly positively ordered, $s^i \neq s^j$ for any nonnegative integers $i \neq j$, and thus the ring $R[[S,\omega]]$ is isomorphic to the skew power series ring $R[[x;\omega]]$. Hence $R[[x;\omega]]$ is right Gaussian, and so Theorem 3.7.2 implies that R is countable-injective strongly regular, establishing the implication (1) $\Rightarrow$ (10) in this case.

We are left with the case where S is not cyclic. By Lemma 4.2.14, in this case in S there exist an element t and a sequence $(s_n)_{n\in\mathbb{N}}$ such that

$$s_1 < s_2 < s_3 < \ldots < t.$$

Hence by Lemma 4.3.3, R is a finite direct proof of division rings, which completes the proof of the implication (1) $\Rightarrow$ (10). $\square$

Chapter 5

Rings of quotients of right Gaussian rings

This Chapter is based on part of:

• R. Mazurek, M. Ziembowski, *Right Gaussian rings and skew power series rings*, submitted.

Rings of quotients are very useful tool in studying commutative Gaussian rings. In the noncommutative setting the situation is different, as for a right Gaussian ring a ring of quotients may not exist (see Example 5.2.1), and even when it exists, it need not be right Gaussian (see Example 5.2.2). The aim of this section is to explain when a ring of right quotients of a right Gaussian ring (if it does exist) is again right Gaussian.

5.1 When is a right ring of quotients of a right Gaussian ring right Gaussian?

In the following lemma we give a sufficient condition for a right ring of quotients of a right Gaussian ring to be again right Gaussian.

Lemma 5.1.1. *Let R be a right Gaussian ring, S a right denominator set in R,* *and R_S a right ring of quotients with respect to S. If for any $a \in R$ we have*

$$Sa \subseteq aS \quad or \quad \exists_{s \in S}\ as = 0, \tag{5.1.1}$$

then R_S is right Gaussian.

Proof. Let $f = \sum_{i=0}^{m} \alpha_i x^i$, $g = \sum_{j=0}^{n} \beta_j x^j \in R_S[x]$, and let $fg = \sum_{k=0}^{m+n} \gamma_k x^k$. We show that if $\delta \in R_S$, then $\alpha_i \delta \beta_j \in \sum_{k=0}^{m+n} \gamma_k R_S$ for all i, j, which obviously implies that R_S is right Gaussian.

Let $\varphi : R \to R_S$ be a ring homomorphism satisfying conditions (a)–(c) of the definition of R_S, recalled in Section 1.2. Since in R_S any finite number of quotients can be brought to a common denominator (see [41, p. 301]), there exist $s \in S$ and $a_0, \ldots, a_m$, $b_0, \ldots, b_n, d \in R$ such that $\alpha_i = \varphi(a_i)\varphi(s)^{-1}$ for any $i \in \{0, \ldots, m\}$, $\beta_j = \varphi(b_j)\varphi(s)^{-1}$ for any $j \in \{0, \ldots, n\}$, and $\delta = \varphi(d)\varphi(s)^{-1}$. Since R is right Gaussian, R is right duo by Lemma 2.2.5, and thus there exist $b'_0, b'_1, \ldots, b'_n, d' \in R$ such that $b_j s = s b'_j$ for any $j \in \{0, \ldots, n\}$, and $ds = sd'$.

Now we consider the polynomials $\hat{f} = \sum_{i=0}^{m} a_i x^i$, $\hat{g} = \sum_{j=0}^{n} b'_j x^j \in R[x]$. Let $\hat{f}\hat{g} = \sum_{k=0}^{m+n} c_k x^k$. Note that for any k,

$$\varphi(c_k) = \sum_{i+j=k} \varphi(a_i)\varphi(b'_j) = \sum_{i+j=k} \varphi(a_i)\varphi(s)^{-1}\varphi(sb'_j) = \sum_{i+j=k} \varphi(a_i)\varphi(s)^{-1}\varphi(b_j s) =$$

$$= \sum_{i+j=k} \varphi(a_i)\varphi(s)^{-1}\varphi(b_j)\varphi(s)^{-1}\varphi(s)^2 = \gamma_k \varphi(s)^2 \in \gamma_k R_S.$$

Note also that since R is right Gaussian, for any i, j we have $a_i d' b'_j \in \sum_{k=0}^{m+n} c_k R$, and thus

$$\varphi(a_i d' b'_j) \in \sum_{k=0}^{m+n} \varphi(c_k) R_S \subseteq \sum_{k=0}^{m+n} \gamma_k R_S. \tag{5.1.2}$$

We are now in a position to show that $\alpha_i \delta \beta_j \in \sum_{k=0}^{m+n} \gamma_k R_S$ for any pair i, j. The case where $\beta_j = 0$ is clear. Thus we assume that $\beta_j \neq 0$, and $\varphi(b_j) \neq 0$ follows. Hence (5.1.1) implies that $Sb_j \subseteq b_j S$, and thus $sb_j = b_j t$ for some $t \in S$. Since

$b_jts = s^2b'_j$ and $ts \in S$, we get

$$\alpha_i\delta\beta_j = \varphi(a_i)\varphi(s)^{-1}\varphi(d)\varphi(s)^{-1}\varphi(b_j)\varphi(s)^{-1} = \varphi(a_i)\varphi(d')\varphi(s)^{-2}\varphi(b_j)\varphi(s)^{-1} =$$

$$= \varphi(a_i)\varphi(d')\varphi(b'_j)\varphi(st)^{-1}\varphi(s)^{-1} \in \varphi(a_id'b'_j)R_S.$$

Applying (5.1.2), we obtain $\alpha_i\delta\beta_j \in \sum_{k=0}^{m+n}\gamma_kR_S$, as desired. □

Theorem 5.1.2. *Let R be a right Gaussian ring, P an ideal of R such that $S = R \setminus P$ is a right denominator set in R, and R_S a right ring of quotients with respect to S. Then the following conditions are equivalent:*

(1) R_S is right Gaussian.

(2) R_S is right duo.

(3) For any $a \in R$ we have $Sa \subseteq aS$ or $\exists_{s\in S}\ as = 0$.

Proof. (1) $\Rightarrow$ (2) follows from Lemma 2.2.5.

(2) $\Rightarrow$ (3) Assume that R_S is right duo, and consider any element $a \in R$ with $Sa \nsubseteq aS$. Then for some $t \in S$ we have $ta \notin aS$. Since R is right Gaussian, R is right duo by Lemma 2.2.5, and we deduce that $ta = ap$ for some $p \in P$. Hence in R_S we have $\varphi(a) = \varphi(t)^{-1}\varphi(ap)$, where $\varphi : R \to R_S$ is a ring homomorphism satisfying conditions (a)–(c) of the definition of R_S, recalled in Section 1.2.. Since R_S is right duo, it follows that $\varphi(a) = \varphi(ap)\alpha$ for some $\alpha \in R_S$. Since P is an ideal of R, R_S is a local ring and the Jacobson radical of R_S is equal to $J(R_S) = \{\varphi(q)\varphi(z)^{-1} : q \in P, z \in S\}$. Hence $\varphi(p)\alpha \in J(R_S)$, and thus $\varphi(a) = \varphi(a)\varphi(p)\alpha \in \varphi(a)J(R_S)$, which implies $\varphi(a) = 0$. Therefore, for some $s \in S$ we have $as = 0$.

(3) $\Rightarrow$ (1) follows from Lemma 5.1.1. □

5.2 Examples

The following example shows that a right Gaussian ring may not have a right ring of quotients with respect to a multiplicative set. In fact the example was constructed by G. Puninski in [66, Section 7] to show that a (right and left) distributive ring R may be not localizable (see also [79, Example 3.3]). Since the ring R in the example is (right and left) duo (see [66, Lemma 7.2(2)]), Theorem 2.2.11 implies that R already is a right Gaussian ring.

Example 5.2.1. Let $Z = \mathbb{Z}[i]$ be the ring of Gaussian integers and $\mathbb{Q}(i)$ the field of fractions of Z. Let $V_1 = Z_{S_1}$ and $V_2 = Z_{S_2}$ (where $S_1 = (2-i)Z$ and $S_2 = (2+i)Z$) be the localization of Z with respect to the maximal ideals $(2-i)Z$ and $(2+i)Z$, respectively, and set $D = V_1 \cap V_2$. Then D is a Prüfer domain and the ideal P of D generated by $2-i$ is a maximal ideal of D. Denote by M the right D-module $\mathbb{Q}(i)/D_S$, where $S = D\backslash P$ and D_S is the localization of D with respect to P, and made M into a left D-module by defining $dm = m\overline{d}$ for any $d \in D$ and $m \in M$, where $\overline{d}$ is the complex conjugation of d. Let R be the set of all matrices of the form $\left(\begin{smallmatrix} d & m \\ 0 & d \end{smallmatrix}\right)$, where $d \in D$ and $m \in M$. Then R is a ring with usual addition and multiplication of matrices as operations. It is proved in [66, Section 7] that R is a right duo right distributive ring (hence R is right Gaussian by Theorem 2.2.11) and $S = R \setminus \{\left(\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right) : m \in M\}$ is a multiplicative subset in R such that R has no right ring of quotients with respect to S. $\square$

We close the chapter with an example of a right Gaussian ring R such that for some ideal P of R, the right ring R_S of quotients with respect to $S = R \setminus P$ exists but is not right Gaussian. Since in the example the set S consists of all regular elements of R, in the same time it is an example of a right Gaussian ring such that the classical right ring of quotients of R exists but is not right Gaussian. In the example we use the skew generalized power series ring which construction is described in Section 4.1.

Example 5.2.2. Let G be the free abelian group generated by the set $\{x_i : i \in \mathbb{N}\}$

and let ψ be an endomorphism of G defined by

$$\psi(x_i) = x_{i+1} \text{ for any } i \in \mathbb{N}.$$

For any $g_1, g_2 \in G$ we write $g_1 \preccurlyeq g_2$ if either $g_1 = g_2$, or $g_1 \neq g_2$ and $g_1^{-1}g_2 = x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$ with $k_n > 0$. It is easy to see that $(G, \preccurlyeq)$ is a totally ordered group and for any $g_1, g_2 \in G$, $g_1 \prec g_2$ implies $\psi(g_1) \prec \psi(g_2)$.

To construct the desired ring R, we first consider the set T of all pairs $(m, g) \in \mathbb{Z} \times G$ such that either $m > 0$, or $m = 0$ and $g \succcurlyeq 1$. We define a multiplication and an order relation in T by setting for $(m_1, g_1), (m_2, g_2) \in T$ that

$$(m_1, g_1)(m_2, g_2) = (m_1 + m_2, \psi^{m_2}(g_1)g_2),$$

and

$$(m_1, g_1) \leqslant (m_2, g_2) \Leftrightarrow \text{ either } m_1 < m_2 \text{ or } m_1 = m_2 \text{ and } g_1 \preccurlyeq g_2.$$

It is easy to verify that $(T, \leqslant)$ is a positively strictly ordered monoid. Furthermore, since the order $\leqslant$ on T is total, and $(m_1, g_1) \leqslant (m_2, g_2)$ implies

$$(m_2, g_2) = (m_1, g_1)(m_2 - m_1, \psi^{m_2 - m_1}(g_1^{-1})g_2) \in (m_1, g_1)T,$$

it follows that T is a right chain monoid.

Let D be a division ring and $\omega : T \to \mathrm{End}(D)$ the trivial monoid homomorphism (i.e. ω_t is the identity map of D for any $t \in T$). Since $(T, \leqslant)$ is a strictly, positively and totally ordered right chain monoid, it follows from Theorem 4.3.1 and [56, Theorem 4.7] that the skew generalized power series ring $R = D[[T, \omega]]$ is a right duo right chain ring. Moreover it is easy to see that R is domain.

Since $(T, \leqslant)$ is a strictly totally ordered monoid, for any $f \in R \setminus \{0\}$ the set $\mathrm{supp}(f)$ is well-ordered and thus there exists a unique minimal element $\pi(f)$ of $\mathrm{supp}(f)$. It is obvious that for any $f, h \in R$, if $f + h \neq 0$ and $\pi(f) \geqslant \pi(h)$, then

$\pi(f+h) \geqslant \pi(h)$. Furthermore, since D is a division ring, for any $f, h \in R$ with $fh \neq 0$ we have

$$\pi(fh) = \pi(f)\pi(h), \tag{5.2.1}$$

and since $(S, \leqslant)$ is positively ordered, it follows that $\pi(fh) \geqslant \pi(f)$ and $\pi(fh) \geqslant \pi(h)$. Now it is clear that the set

$$I = \{0\} \cup \{f \in R \setminus \{0\} : \pi(f) > (1, x_1^i x_2^j x_3) \text{ for any } i, j \in \mathbb{Z}\}$$

is an ideal of R. In what follows the "bars" refer to modulo I, that is $\overline{R}$ stands for the factor ring R/I, and $\bar{f} = f + I$ for any $f \in R$. Since R is a right duo right chain ring, so is $\overline{R}$, and thus $\overline{R}$ is a right Gaussian ring by Corollary 2.2.16. Furthermore, since in any right chain ring the set of right zero-divisors as well as the set of left zero-divisors are ideals (see [9, Lemma 2.3(i)]), the set P of those elements of $\overline{R}$ that are right or left zero-divisors is an ideal of $\overline{R}$, and $S = \overline{R} \setminus P$ is a right denominator set in $\overline{R}$. Hence a right ring of quotients $\overline{R}_S$ exists; note that since S coincides with the set of regular elements of $\overline{R}$, $\overline{R}_S$ is the classical ring of quotients of $\overline{R}$.

By Theorem 5.1.2, to show that $\overline{R}_S$ is not right Gaussian it suffices to prove that $\bar{\mathsf{e}}_{(0,x_1)} \in S$ and $\bar{\mathsf{e}}_{(0,x_1)}\bar{\mathsf{e}}_{(1,1)} \notin \bar{\mathsf{e}}_{(1,1)}S$. We first prove that $\bar{\mathsf{e}}_{(0,x_1)} \in S$. Otherwise $\bar{\mathsf{e}}_{(0,x_1)}$ is a left or right zero-divisor in $\overline{R}$, and thus $\bar{\mathsf{e}}_{(0,x_1)}\bar{f} = 0$ or $\bar{f}\bar{\mathsf{e}}_{(0,x_1)} = 0$ for some $f \in R \setminus I$. In the first case we have $\mathsf{e}_{(0,x_1)}f \in I$, and using (5.2.1) we obtain that for any $i, j \in \mathbb{Z}$,

$$(0, x_1)\pi(f) = \pi(\mathsf{e}_{(0,x_1)})\pi(f) = \pi(\mathsf{e}_{(0,x_1)}f) > (1, x_1^i x_2^j x_3).$$

Hence $\pi(f) > (0, x_1)^{-1}(1, x_1^i x_2^j x_3) = (0, x_1^{-1})(1, x_1^i x_2^j x_3) = (1, x_1^i x_2^{j-1} x_3)$ for any $i, j \in \mathbb{Z}$, which implies that $f \in I$, a contradiction. Analogously one obtains a contradiction in the second case, and thus $\bar{\mathsf{e}}_{(0,x_1)} \in S$.

Now we show that $\bar{\mathsf{e}}_{(0,x_1)}\bar{\mathsf{e}}_{(1,1)} \notin \bar{\mathsf{e}}_{(1,1)}S$. For a contradiction, suppose that $\bar{\mathsf{e}}_{(0,x_1)}\bar{\mathsf{e}}_{(1,1)} = \bar{\mathsf{e}}_{(1,1)}\overline{f}$ for some $f \in R$ such that $\overline{f} \in S$. Then, since $\mathsf{e}_{(0,x_1)}\mathsf{e}_{(1,1)} = \mathsf{e}_{(1,x_2)}$, it follows that $\mathsf{e}_{(1,x_2)} - \mathsf{e}_{(1,1)}f \in I$. This and the definition of I imply that if $\pi(\mathsf{e}_{(1,x_2)}) \neq \pi(\mathsf{e}_{(1,1)}f)$, then $\mathsf{e}_{(1,x_2)} \in I$, a contradiction. Hence $(1,x_2) = \pi(\mathsf{e}_{(1,x_2)}) = \pi(\mathsf{e}_{(1,1)}f) = (1,1)\pi(f)$, and thus $\pi(f) = (0,x_2)$. Therefore, for any $i,j \in \mathbb{Z}$ we have

$$\pi(f\mathsf{e}_{(1,x_3)}) = \pi(f)\pi(\mathsf{e}_{(1,x_3)}) = (0,x_2)(1,x_3) = (1,x_3^2) > (1,x_1^i x_2^j x_3),$$

which shows that $f\mathsf{e}_{(1,x_3)} \in I$. Hence $\overline{f}\overline{\mathsf{e}}_{(1,x_3)} = 0$ in $\overline{R}$, and since $\overline{\mathsf{e}}_{(1,x_3)} \neq 0$, it follows that $\overline{f}$ is a left zero-divisor of $\overline{R}$. Thus $\overline{f} \notin S$, and this contradiction completes our proof that the ring $\overline{R}_S$ is not right Gaussian. $\square$

Chapter 6

Homomorphic images of polynomials rings

As the main part of the present chapter we consider a class of homomorphic images of a polynomial ring $R[x]$ and give the necessary and sufficient conditions for a ring R under which these images are right Gaussian. We will also say a little about the trivial ring extension in the context of right Gaussian rings.

6.1 Polynomials ring

Remark 6.1.1. In [2] Anderson and Camillo obtained, among many other things, the result which says that for a ring R and every integer $n \geqslant 2$, $R[x]/(x^n)$ (recall that (x^n) is the ideal of $R[x]$ generated by x^n is an Armendariz ring if and only if R is a reduced ring.

The our next goal is to obtain the necessary and sufficient conditions for a ring R, under which the ring $A = R[x]/(x^n)$ is right Gaussian.

First of all we have the following (see [24]):

Lemma 6.1.2. *A ring R is strongly regular if and only if every homomorphic image of R is a reduced ring and R is right duo.*

Proof. For the "only if" part let us assume that I is an ideal of R and $r \in R$.

Since R is strongly regular, there exists $x \in R$ such that $r = r^2x$. If for $\overline{r} \in R/I$, $\overline{r}^2 = 0$, then $r^2 \in I$. Thus $r = r^2x \in I$ and therefore $\overline{r} = 0$. So R/I is reduced. The fact that R is right duo follows from Lemma 2.2.14.

Let $r \in R$. If R is right duo, then r^2R is an ideal of R. It is clear that for the element $\overline{r} \in R/r^2R$, $\overline{r}^2 = 0$. Since by assumption R/r^2R is reduced ring, $\overline{r} = 0$ what implies that $r \in r^2R$. Thus by definition, R is strongly regular and the "if" part is completed. □

To prove the main result of the chapter we will need the following:

Lemma 6.1.3. *Let R be a strongly regular ring and $n > 1$. Then for every $f = \sum_{i=0}^{n-1} a_ix^i + (x^n) \in R[x]/(x^n)$, there exist central mutually orthogonal idempotents $e_0, ..., e_{n-1} \in R$ such that $f \cdot R[x]/(x^n) = g \cdot R[x]/(x^n)$ for $g = \sum_{i=0}^{n-1} e_ix^i + (x^n) \in R[x]/(x^n)$. In this case the ring $R[x]/(x^n)$ is right duo.*

Proof. Let $A = R[x]/(x^n)$ and $f = \sum_{i=0}^{n-1} a_ix^i + (x^n) \in A$. Since R is strongly regular by Lemma 2.2.14 there exist central idempotents $f_0, ..., f_{n-1} \in R$ and units $u_1, ..., u_{n-1} \in R$ such that $a_i = u_if_i$ for every $i \in \{0, \ldots, n-1\}$.

Let $e_0 = f_0$ and $e_i = f_i(1 - f_{i-1})...(1 - f_0)$ for every $i \in \{1, \ldots, n-1\}$. It is easy to see that $e_0, ..., e_{n-1}$ are central mutually orthogonal idempotents of R. We claim that $fA = gA$ for $g = \sum_{i=0}^{n-1} e_ix^i + (x^n) \in A$.

Let for every $a \in R[x]$, $\overline{a} = a + (x^n) \in A$. Notice that $\overline{a_0} = \overline{f_0}\overline{u_0} = \overline{e_0u_0} = g\overline{e_0u_0} \in gA$. Let $0 < i \leqslant n-1$ and assume that $\overline{a_0}, \overline{a_1x}, ..., \overline{a_{i-1}x^{i-1}} \in gA$. Then $\overline{f_0}, \overline{f_1x}, ..., \overline{f_{i-1}x^{i-1}} \in gA$ and we deduce that there exists an element $k \in gA$ such that

$$g\overline{e_iu_i} = \overline{e_iu_ix^i} = \overline{f_i(1 - f_{i-1})...(1 - f_0)u_ix^i} = \overline{f_iu_ix^i} + k = \overline{a_ix^i} + k.$$

Thus $\overline{a_ix^i} \in gA$ and we deduce that $fA \subseteq gA$.

Now we want to show that $gA \subseteq fA$. For this we notice that for every $i \in$

$\{1, ..., n-1\}$

$$f\overline{e_i} = \overline{e_i x^i} \cdot \overline{u_i} \cdot \overline{(1 + u_i^{-1} a_{i+1} x + ... + u_i^{-1} a_{n-1} x^{n-i-1})}.$$

Since it is easy to see that the element $\overline{u_i} \cdot \overline{(1 + u_i^{-1} a_{i+1} x + ... + u_i^{-1} a_{n-1} x^{n-i-1})}$ is invertible in A, we conclude that $\overline{e_i x^i} \in fA$, and finally $gA \subseteq fA$. So we have proved that $fA = gA$.

It remains to show that A is right duo. For that let $f \in A$ and let g be as above. Then $fA = gA$ and for every $w = \sum_{i=0}^{n-1} w_i x^i + (x^n) \in A$ we have $wf = wgv$ for some $v \in A$. Moreover since $e_0, \ldots, e_{n-1}$ are central mutually orthogonal idempotents we have $wg = gw$. Thus $wf = wgv = gwv \in gA = fA$, so the fact that A is right duo follows. □

Now we are in a position to prove the main result of the chapter.

Theorem 6.1.4. *Let R be a ring. Then for every $n > 1$ the following conditions are equivalent:*

(1) *$R[x]/(x^n)$ is right Gaussian.*

(2) *$R[x]/(x^n)$ is right distributive.*

(3) *R is strongly regular.*

Proof. Set $A = R[x]/(x^n)$.

(1) $\Rightarrow$ (3) If A is right Gaussian, then A is right duo by Lemma 2.2.5, and it is easy to see that R is right duo as well. Let I be an ideal of R and $\overline{R} = R/I$. Then $\overline{R}[x]/(x^n)$ is an homomorphic image of $R[x]/(x^n)$, so the assumption and Theorem 2.2.6 imply the fact that $\overline{R}[x]/(x^n)$ is Armendariz. Thus by Corollary 8.4.6 from Chapter 8, $\overline{R}$ is reduced. Now Lemma 6.1.2 implies that R is strongly regular.

(3) $\Rightarrow$ (2) By Lemma 6.1.3, $R[x]/(x^n)$ is right duo. Hence by Proposition 3.1.3(ii) it is enough to show that A is right Bézout. Let $a, b \in A$. Then by Lemma 6.1.3

there exist

$$g = \sum_{i=0}^{n-1} e_i x^i + (x^n) \in A$$

where $e_0, ..., e_{n-1}$ are mutually orthogonal idempotents of R and

$$p = \sum_{i=0}^{n-1} f_i x^i + (x^n) \in A$$

where $f_0, ..., f_{n-1}$ are mutually orthogonal idempotents of R as well, such that $aA = gA$ and $bA = pA$. Now we consider the element

$$h = (e_0 + f_0(1-e_0)) + \sum_{i=1}^{n-1} [e_i(1-f_0)...(1-f_{i-1}) + f_i(1-e_0)...(1-e_i)]x^i + (x^n) \in A.$$

Let for every $a \in R[x]$, $\overline{a} = a + (x^n) \in A$. Then for h we have $\overline{e_0} = h\overline{e_0} \in hA$ and $\overline{f_0} = h\overline{f_0} \in hA$. Now let us assume that for some $0 \leqslant i < n-1$, $\overline{e_0}, \overline{f_0}, ..., \overline{e_i x^i}, \overline{f_i x^i} \in hA$. Then

$$h\overline{e_{i+1}} = \overline{f_0 e_{i+1}} + \overline{f_1 e_{i+1} x} + ... + \overline{f_i e_{i+1} x^i} + \overline{e_{i+1}(1-f_0)...(1-f_i)x^{i+1}}.$$

Since $\overline{e_0}, \overline{f_0}, ..., \overline{e_i x^i}, \overline{f_i x^i} \in hA$ we deduce that $\overline{e_{i+1} x^{i+1}} \in hA$.

Now we notice that

$$h\overline{f_{i+1}} = \overline{e_0 f_{i+1}} + \overline{e_1 f_{i+1} x} + ... + \overline{e_i f_{i+1} x^i} + \overline{e_{i+1} f_{i+1} x^{i+1}} + \overline{f_{i+1}(1-e_0)...(1-e_{i+1})x^{i+1}}$$

what implies that $\overline{f_{i+1} x^{i+1}} \in hA$ since $\overline{e_0}, \overline{f_0}, ..., \overline{e_i x^i}, \overline{f_i x_i}, \overline{e_{i+1} x^{i+1}} \in hA$.

Thus we conclude that

$$\overline{e_0}, \overline{f_0}, ..., \overline{e_{n-1} x^{n-1}}, \overline{f_{n-1} x^{n-1}} \in hA$$

and $gA + pA \subseteq hA$ follows. Since this is obvious that $hA \subseteq gA + pA$, the ring A is right Bézout.

(2) $\Rightarrow$ (1) Since $R[x]/(x^n)$ is right distributive and $n > 1$, by Theorem 2.1.2 for every $a \in R$ there exist $f = \sum_{i=0}^{n-1} f_i x^i + (x^n), q = \sum_{i=0}^{n-1} q_i x^i + (x^n)$, $k \in A$ such

that $af = xk$ and $x(1-f) = aq$. Then $af_0 = 0$ and $1 - f_0 = aq_1$, what implies that $a = a(1-f_0) = a^2q_1$. Thus R is strongly regular. Now Lemma 6.1.3 implies that A is right duo, so by Theorem 2.2.11 the ring A is right Gaussian. $\square$

Corollary 6.1.5. *If R is a commutative ring, then the following conditions are equivalent:*

(1) $R[x]$ is Gaussian.

(2) $R[x]/(x^n)$ is Gaussian.

(3) R is von Neumann regular.

Proof. (1) $\Rightarrow$ (2) It follows from Proposition 2.2.2

(2) $\Rightarrow$ (3) By Theorem 6.1.4 the ring R is strongly regular what for commutative rings means exactly that R is von Neumann regular.

(3) $\Rightarrow$ (1) Follows from Proposition 2.2.7. $\square$

Remark 6.1.6. A different situation than that in Corollary 6.1.5 holds for noncommutative rings. Namely by Proposition 2.2.7, $R[x]$ is right Gaussian if and only if R is commutative von Neumann regular. On the other hand we have proved that $R[x]/(x^n)$ is right Gaussian if and only if R is strongly regular. Thus if R is a noncommutative strongly regular ring, then $R[x]$ is not right Gaussian but for every $n > 1$ its homomorphic image $R[x]/(x^n)$ is right Gaussian.

6.2 Trivial ring extension

In [5] among other things the authors consider the transfer of properties which define commutative Gaussian rings, between a commutative ring and the trivial ring extension. The main result in this context which appears there is [5, Theorem 2.1]. The present chapter provides information about possibilities to get similar facts for noncommutative rings. The following generalizes [5, Theorem 2.1(1)].

Proposition 6.2.1. *Let R be a ring. If for a left and right R-module M the trivial ring extension $A = R \propto M$ is right Gaussian, then R is right Gaussian.*

Proof. Since A is right Gaussian for every $a, b \in R$ there exist $c \in R$ and $m \in M$ such that $(ba, 0) = (b, 0)(a, 0) = (a, 0)(c, m) = (ac, am)$. Then $ba = ac$ and the fact that R is right duo follows. It is easy to see that $\varphi : A \to R$ such that $\varphi(a, m) = a$ is a ring epimorphism. It is obvious that if B is a homomorphic image of the ring R, then B is homomorphic image of A as well. Hence B has to be Armendariz ring and then Theorem 2.2.6 implies that R is right Gaussian. □

The second part of the [5, Theorem 2.1] says that for a commutative local ring R with unique maximal ideal of R equal to $J(R)$, for the trivial ring extension $A = R \propto R/J(R)$, we have the converse of the fact presented in Proposition 6.2.1. Namely, the ring $A = R \propto R/J(R)$ is Gaussian if and only if so is R. It turns out that it is not the case for noncommutative local rings as the next example shows.

Example 6.2.2. Let K be a field and $D = K((x, y))$ the field of rational functions with two variables. Let $\sigma : D \to D$ be the isomorphism of D such that $\sigma(x) = y$, $\sigma(y) = x$ and $\sigma(k) = k$ for every $k \in K$. Then the ring $R = D[[X; \sigma]]$ is local and $J(R) = \{f = \sum_{i=0}^{\infty} a_i X^i \in R : a_0 = 0\}$, where $J(R)$ denotes the Jacobson radical of R. Moreover by Theorem 3.7.2 the ring R is right Gaussian. Let $A = R \propto R/J(R)$ and for every $r \in R$, $\overline{r}$ denotes the image of R in $R/J(R)$. Then for every $(\alpha, \beta) \in A$

$$(x, 0)(X, \overline{x}) = (xX, \overline{x}^2),$$

$$(X, \overline{x})(\alpha, \beta) = (X\alpha, \overline{x\alpha}).$$

If $(x, 0)(X, \overline{x}) = (X, \overline{x})(\alpha, \beta)$, then $xX = X\alpha = \sigma(\alpha)X$, and we get $x = \sigma(\alpha)$ what implies that $\alpha = y$. But then $\overline{x}^2 = \overline{x\alpha} = \overline{xy}$, so $x^2 - xy \in J(R)$, a contradiction. Thus we have proved that A is not right duo, so the fact that A is not right Gaussian follows from Lemma 2.2.5. □

Remark 6.2.3. It is easy to see that for every ring R the ring $R[x]/(x^2)$ is isomorphic to the trivial ring extension $R \propto R$. So, in particular, in the previous section we described right Gaussian trivial ring extension $R \propto R$.

Chapter 7

Noncommutative Prüfer rings

In this chapter we will consider relation between right Gaussian rings and noncommutative Prüfer rings. The present chapter also includes questions which can be a direction to move on further study.

7.1 On noncommutative Prüfer rings

At the beginning of the chapter we want to recall the definition of Prüfer domains.

Definition 7.1.1. A commutative domain D is called a *Prüfer domain* if for every finitely generated ideal I of D, $I \cdot I^{\star} = D$, where $I^{\star} = \{q \in Q_{cl}(D) : qI \subseteq D\}$ and $Q_{cl}(D)$ is the field of fractions of D.

It is well known that the following theorem is true. One can see it using results from [22].

Theorem 7.1.2. *If a ring R is commutative domain, then the following conditions are equivalent:*

(1) R is semihereditary.

(2) R has weak dimension less than or equal to one.

(3) R is distributive.

(4) R is Gaussian.

(5) R is Prüfer domain.

In [1], J.H Alajbegovic and N.I. Dubrovin indroduced the concept of noncommutative right Prüfer rings, which are a generalization of commutative Prüfer domains. Regarding Theorem 7.1.2 it seems to be interesting to ask if there holds analogous result for noncommutative rings. In present chapter we will deal with this issue. But first of all we want to define noncommutative right Prüfer rings (see [1, Definition 1.1])

Let R be a prime Goldie ring with classical ring of quotients $Q_{cl}(R)$. Recall that an additive subgroup I of $Q_{cl}(R)$ is a *right R-ideal* provided (i) $IR \subseteq I$, (ii) I contains a regular element of $Q_{cl}(R)$, and (iii) $dI \subseteq R$ for a regular element $d \in Q_{cl}(R)$.

Definition 7.1.3. ([1]) A prime Goldie ring R is called a *right Prüfer ring* if every finitely generated right R-ideal I satisfies the equalities

$$I^{-1} \cdot I = R, \quad I \cdot I^{-1} = O_l(I), \tag{7.1.1}$$

where $O_l(I) = \{q \in Q_{cl}(R) : qI \subseteq I\}$ and $I^{-1} = \{q \in Q_{cl}(R) : IqI \subseteq I\}$.

Obviously in an appropriate way we can define left Prüfer rings. We have the following:

Proposition 7.1.4. *A prime Goldie ring R is a right Prüfer ring if and only if it is a left Prüfer ring.*

Proof. See [1, Proposition 1.12] □

Now we want to show that there exists a ring R which is right Prüfer, right distributive, right semihereditary and has weak dimension less than or equal to one but is not right duo (see [56, Example 5.11]), which implies that R is not right Gaussian. There is a reason why in our investigations connected with Theorem 7.1.2 in noncommutative setting, we will assume that R is right duo. To present the example we will need the following:

Proposition 7.1.5. *If a ring R is right Prüfer, then R is right and left semihereditary.*

Proof. See [1, Proposition 1.8]. $\square$

Example 7.1.6. There exists a ring which is right Prüfer, right distributive, left and right semihereditary and has weak dimension less than or equal to one, but is not right Gaussian.

Proof. Let $F = K((x))$ be the field of Laurent series over an ordered field K. The field F is ordered by

$$k_i x^i + k_{i+1}x^{i+1} + k_{i+2}x^{i+2} + \cdots > 0 \Leftrightarrow k_i > 0.$$

We order lexicographically the set

$$S = \{(a, b) \in F \times F : \text{ either } a > 1, \text{ or } a = 1 \text{ and } b \geqslant k \text{ for some } k \in K\}$$

and define an operation in S by

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1a_2 + b_2),$$

obtaining a strictly totally ordered monoid $(S, \cdot, \leqslant)$. Since for any $s, t \in S$, $s \leqslant t$ implies $St \subseteq Ss$ and $tS \subseteq sS$, S is a chain monoid. Furthermore, each $s \in S$ with $s \leqslant 1$ is invertible in S, and thus by [56, Theorem 4.7], [56, Theorem 5.10] and [56, Lemma 4.2], for any division ring D and the trivial monoid homomorphism $\omega : S \to \mathrm{End}(D)$ (i.e., ω_s is the identity map of D for any $s \in S$) the ring $A = D[[S, \omega]]$ is a chain domain. Since $sS \subseteq Ss$ for any $s \in S$, and by [56, Theorem 5.10] each $f \in A \setminus \{0\}$ can be written in the form ve_s with $s \in S$ and $v \in U(A)$, it follows that A is a left duo domain. However, A is not right duo, since $e_{(1,-1)}e_{(x^{-1},0)} \notin e_{(x^{-1},0)}A$. Indeed, otherwise $e_{(x^{-1},-x^{-1})} = e_{(1,-1)}e_{(x^{-1},0)} = e_{(x^{-1},0)}g$ for some $g \in A$, and we obtain $(e_{(x^{-1},0)}g)((x^{-1}, -x^{-1})) = e_{(x^{-1},-x^{-1})}((x^{-1}, -x^{-1})) = 1$. Hence there exists

$(a, b) \in \mathrm{supp}(g)$ such that $(x^{-1}, -x^{-1}) = (x^{-1}, 0)(a, b)$. The only possibility is $(a, b) = (1, -x^{-1})$, but $(1, -x^{-1}) \notin S$, a contradiction. Now it is obvious that A is not right duo but is a chain domain.

The ring A, being a chain domain, is a prime Goldie ring which is distributive and Bézout. Thus [1, Example 1.15] implies that A is right Prüfer. By Proposition 7.1.5 the ring A is also left and right semihereditary. Since A is left duo, the left version of Theorem 3.6.6 implies that A has weak dimension less than or equal to one. Since A is not right duo, by Lemma 2.2.5 A is not right Gaussian. □

7.2 Conection between some classes of rings

By Theorem 2.2.11 if R is a right duo right distributive ring, then R is right Gaussian. Proposition 7.2.2 shows that for some classes of rings the converse holds.

Lemma 7.2.1. *Let R be a right Gaussian ring, and let $a, b, \alpha, \beta \in R$ be such that*

$$\alpha a = \beta b \quad \text{and} \quad r_R(\beta) = \{0\}.$$

Then there exists $s \in R$ such that $as \in bR$ and $b(1-s) \in aR$.

Proof. Since in $R[x]$ we have $(\alpha+\beta x)(b-ax) = \alpha b - \beta a x^2$ and R is right Gaussian, it follows that

$$\alpha a = \alpha b p + \beta a q \quad \text{for some} \quad p, q \in R. \tag{7.2.1}$$

Since in $R[x]$ we have $(\alpha - \beta x)(bp - a + apx) = -\beta a q + \beta a x - \beta a p x^2$ and R is right Gaussian, it follows that $\beta b p = \alpha a p \in \beta a R$, and since $r_R(\beta) = \{0\}$, we obtain $bp \in aR$. Hence $bp = as$ for some $s \in R$. Now from (7.2.1) we obtain $\beta b(1-s) = \beta a q$, and since $r_R(\beta) = \{0\}$, we deduce that $b(1-s) = aq$. Hence $as = bp \in bR$ and $b(1-s) = aq \in aR$. □

Recall that a ring R is *left uniform* if $Ra \cap Rb \neq 0$ for any $a, b \in R \setminus \{0\}$.

Proposition 7.2.2. *If R is a left uniform right Gaussian domain, then R is right distributive.*

Proof. To prove that R is right distributive, it suffices to show that for any $a, b \in R$ there exist $c, d \in R$ such that

$$c + d = 1, \quad ac \in bR \quad \text{and} \quad bd \in aR. \tag{7.2.2}$$

If $a = 0$, then (7.2.2) holds with $c = 1$ and $d = 0$. Also the case $b = 0$ is clear. We are left with the case where $a \neq 0$ and $b \neq 0$. Then, since R is a left uniform domain, there exist $\alpha, \beta \in R$ with $\alpha a = \beta b$ and $r_R(\beta) \neq 0$. By Lemma 7.2.1, for some $s \in R$ we have $as \in bR$ and $b(1-s) \in aR$. Thus in this case (7.2.2) holds with $c = s$ and $d = 1 - s$. □

As an immediate consequence of the above result we get the following:

Corollary 7.2.3. *Let R be a prime Goldie ring which is right Gaussian, then R is right distributive.*

Proof. Let us assume that R is a prime Goldie ring which is right Gaussian. Then R is right duo, and being prime is domain. Thus R is Goldie domain, and it implies that R is left uniform. Now the fact that R is right distributive follows from Proposition 7.2.2. □

Now we are in a position to prove the following:

Theorem 7.2.4. *Let R be a prime Goldie ring. Then the following conditions are equivalent:*

(1) R is right semihereditary and right duo.

(2) R has weak dimension less than or equal to one and R is right duo.

(3) R is right distributive and right duo.

(4) R is right Gaussian.

Proof. (1) $\Rightarrow$ (2) It follows from Theorem 3.6.6.

(2) $\Rightarrow$ (3) By [76, Lemma 12] if a ring R is semiprime right duo and has weak dimension of R less then or equal to one, then R is right distributive. So the implication is clear.

(3) $\Rightarrow$ (4) It follows from Theorem 2.2.11.

(4) $\Rightarrow$ (1) By Lemma 2.2.5 and Corollary 7.2.3 the ring R is right distributive and right duo. Thus [75, Lemma 2] implies that R is right semihereditary and right duo. $\square$

The above proposition implies that if R is right Prüfer and right duo ring, then all conditions which appear in Theorem 7.2.4 are satistied. Unfortunately we do not know if then we have an equivalence. On the other hand we will prove below that we do have equivalence under some assumptions.

First of all we have to prove the following two lemmas:

Lemma 7.2.5. *Let R be a prime Goldie ring. Moreover, let R be a right semihereditary domain and I be finitely generated right R-ideal. If for every finitely generated right R-ideal J we have $J^{-1}J = R$, then $II^{-1} = O_l(I)$.*

Proof. By [1, Lemma 1.3] for every finitely generated right R-ideal J we have $O_r(J) = R$.

Since I is a right R-ideal, there exist $0 \neq d \in Q$ such that $dI \subseteq R$. This is obvious that then dI a is finitely generated right ideal of R (since I is a right R-ideal, by definition $IR \subseteq I$). So dI is a projective R-module. Hence by [70, Lemma 1.2], $dI(dI)^{-1} = O_l(dI)$.

Notice that

$$x \in (dI)^{-1} \Leftrightarrow dIxdI \subseteq dI \Leftrightarrow IxdI \subseteq I \Leftrightarrow xd \in I^{-1} \Leftrightarrow x \in I^{-1}d^{-1}. \qquad (7.2.3)$$

Therefore we have $(dI)^{-1} = I^{-1}d^{-1}$. Moreover,

$$x \in O_l(dI) \Leftrightarrow xdI \subseteq dI \Leftrightarrow d^{-1}xdI \subseteq I \Leftrightarrow$$

$$\Leftrightarrow d^{-1}xd \in O_l(I) \Leftrightarrow x \in dO_l(I)d^{-1}.$$

Hence we have $O_l(dI) = dO_l(I)d^{-1}$ and it follows that we also have $d^{-1}O_l(dI)d = O_l(I)$. Now notice that we get

$$dI(dI)^{-1} = O_l(dI) \Rightarrow dII^{-1}d^{-1} = O_l(dI) \Rightarrow II^{-1} = d^{-1}O_l(dI)d = O_l(I).$$

So $II^{-1} = O_l(I)$ □

Lemma 7.2.6. *Let R be a prime Goldie ring. If for every finitely generated right ideal J of R such that J includes a regular element of $Q_{cl}(R)$, we have $J^{-1}J = R$, then for every finitely generated right R-ideal I, $I^{-1}I = R$.*

Proof. Let I be a right R-ideal. Then there exists a regular element $d \in Q = Q_{cl}(R)$ such that $dI \subseteq R$. It is easy to see that $dI <_r R$, dI is finitely generated and includes regular element of Q. Hence by assumption

$$(dI)^{-1}(dI) = R \tag{7.2.4}$$

By (7.2.3) we have $(dI)^{-1} = I^{-1}d^{-1}$. Thus, using equation (7.2.4) we get $R = (dI)^{-1}(dI) = I^{-1}d^{-1}dI = I^{-1}I$. □

Now we are able to present the following:

Proposition 7.2.7. *Let R be a prime Goldie ring such that for every element $s \in R \backslash J(R)$, $sR \subseteq Rs$. Then the following conditions are equivalent:*

(1) R is right semihereditary and right duo.

(2) R has weak dimension less than or equal to one and R is right duo.

(3) R is right distributive and right duo.

(4) R is right Gaussian.

(5) R is right Prüfer ring and right duo.

Proof. By Proposition 7.1.5 and Theorem 7.2.4 to complete the proof it is enough to show that (3) implies (5).

For that let us assume that R satisfies (3). Since R is a right duo and prime Goldie ring, it is easy to see that R is a domain. Now we will show that R is a right Prüfer ring.

Let J be a finitely generated right ideal of R such that J includes a regular element of $Q_{cl}(R)$. Then $J = b_1R + ... + b_mR$ for some $b_1, ..., b_m \in R\backslash\{0\}$ and positive integer m.

Let M be a maximal right ideal of R. Since R is right distributive and right duo, using Theorem 2.2.11 we deduce that there exists $s \in R\backslash M$ and $k \in \{1, ..., m\}$ such that

$$b_is \in b_kR, \text{ for every } i \in \{1, ..., m\}. \tag{7.2.5}$$

Let $i \in \{1, ..., m\}$. By (7.2.5) there exists $a_i \in R$ such that $b_is = b_ka_i$. So in $Q_{cl}(R)$

$$sb_k^{-1}b_i = sa_is^{-1}. \tag{7.2.6}$$

Since $s \notin M$, $s \notin J(R)$. Thus by assumption and the fact that R is right duo, $sR = Rs$. Thus the equation (7.2.6) implies that for some $a_i' \in R$, $sb_k^{-1}b_i = sa_is^{-1} = a_i'ss^{-1} = a_i' \in R$. Hence $sb_k^{-1}J \subseteq R$. So $sb_k^{-1} \in [R : J]_l = \{q \in Q_{cl}^r(R) : qJ \subseteq R\}$. Now, notice that $s = sb_k^{-1}b_k \in [R : J]_l \cdot J$.

Up to this point we have showed that for every maximal right ideal M of R, there exists $s \in R\backslash M$ such that $s \in [R : J]_l \cdot J$. Since by definition $[R : J]_l \cdot J \subseteq R$, and it is easy to see that $[R : J]_l \cdot J$ is a right ideal of R, we deduce that $[R : J]_l \cdot J = R$.

By [1, Lemma 1.3], $J^{-1}J = R$, so Lemma 7.2.6 implies that for every finitely generated right R-ideal I, $I^{-1}I = R$. Moreover, since (3) and (1) are equivalent by Theorem 7.2.4, using Lemma 7.2.5 we get the fact that $II^{-1} = O_l(I)$ for every finitely generated right R-ideal I. Thus R is a Prüfer ring. $\square$

Now, we have the following:

Corollary 7.2.8. *Let R be a prime Goldie duo ring. Then the following conditions are equivalent:*

(1) R is semihereditary.

(2) R has weak dimension less than or equal to one.

(3) R is distributive.

(4) R is Gaussian.

(5) R is Prüfer ring.

Now we must pose the following:

Question 7.2.9. *Is it true that if R is a prime Goldie ring which is right Gaussian, then R is Prüfer ring?*

7.3 On semi-Prüfer rings and coincidence of right distributivity and right Gaussianess

The next result gives us another class of rings for which right distributivity and right Gaussianess coincide.

Theorem 7.3.1. *Let R be a left duo reduced ring. Then the following conditions are equivalent:*

(1) R is a right distributive and right duo.

(2) R is a right Gaussian.

Proof. (1) $\Rightarrow$ (2) Obvious by Theorem 2.2.11.

(2) $\Rightarrow$ (1) First of all, we will show that

$$\text{for every } a, b \in R, \text{ if } ab = 0 \text{ then } as = 0 \text{ and } b(1-s) = 0 \text{ for some } s \in R. \tag{7.3.1}$$

For that, let $a, b \in R$ be such that $ab = 0$. To prove (7.3.1) we adopt some ideas of the proof of [21, Lemma 2.1]. First note that

$(*)$ If it is satisfied (2), then $(a^2, b^2, ba-ab)_r = (ab, a^2+b^2, ba)_r$ for any $a, b \in R$.

Indeed, since

$$(a+bx)(a-bx) = a^2 - abx + bax - b^2x^2 = a^2 + (ba-ab)x - b^2x^2,$$

we have $(a, b)_r^2 = (a^2, b^2, ba-ab)_r$. On the other hand, $(a+bx)(b+ax) = ab + (a^2+b^2)x + bax^2$ implies that $(a, b)_r^2 = (ab, a^2+b^2, ba)_r$ and $(*)$ follows.

By $(*)$ we have $(a^2, b^2, ba-ab)_r = (ab, a^2+b^2, ba)_r$, and since R is reduced (reduceness implies that $ba = 0$)

$$b^2 = (a^2+b^2)s \text{ for some } s \in R. \tag{7.3.2}$$

Since $ab = 0$ and R being right Gaussian is right duo, by multiplying (7.3.2) by a from the left, we obtain $0 = a^3s$ and for some $s', s'', v \in R$,

$$(as)^3 = asasas = asa^2s's = a^3s''s's = a^3sv = 0.$$

Since R is reduced, $as = 0$ follows. Similarly, by multiplying (7.3.2) by b, we obtain $b^3 = b^3s$. Thus $b^3(1-s) = 0$, hence $(b(1-s))^3 = 0$, and the reduceness of R implies that $b(1-s) = 0$.

For the rest of the implication we apply Theorem 2.2.11. Let $a, b \in R$, and let M be a maximal right ideal of R.

Since R is left duo there exists an element $a' \in R$ such that

$$a'b = ba.$$

Using a similar argumantation as in the justification of $(*)$ we can show that

$$(a'b, a'a + b^2)_r = (a'a, b^2)_r.$$

Therefore for some $p, r \in R$ we have $b^2 = a'br + (a'a + b^2)p$ which, for $q = p - 1$, can be rewritten in the form

$$a'ap + a'br + b^2q = 0 \tag{7.3.3}$$

and $p \notin M$ or $q \notin M$.

Case 1. Assume that $p \notin M$. Since R is right Gaussian and using (7.3.3) we get

$$(a' - bx)[ap + br + bpx] = -b^2q - b^2rx - b^2px^2,$$

it follows that $bap \in (b^2q, b^2r, b^2p)_r \subseteq (b^2)_r$, and thus $bap = b^2w$ for some $w \in R$. Since $b(ap - bw) = 0$ by (7.3.1), for some $t \in R$ we have $(ap - bw)t = 0$ and $b(1-t) = 0$. If $t \notin M$, then since R is right duo $pt \notin M$ and $a(pt) = bwt \in bR$. If $t \in M$, then $1 - t \notin M$ and $b(1-t) = 0 \in aR$. Hence always there exists $s \in R \setminus M$ such that $as \in bR$ or $bs \in aR$.

Case 2. Assume that $q \notin M$. Then we obtain

$$(b - a'x)(bq + ar + aqx) = -a'ap - a'arx - a'aqx^2$$

and similarly as in the Case 1, for some $m \in R$ we get $a'(bq - am) = 0$. Thus, there exists $t \in R$ such that $(bq - am)t = 0$ and $a'(1-t) = 0$. If $t \notin M$, then $qt \notin M$ and $b(qt) = amt \in aR$ and the proof is complete. In turn, if $(1-t) \notin M$, then we have to notice that since R is left duo, for some $\overline{b} \in R$ we have $\overline{b}a' = a'b$. Thus we obtain $ba(1-t) = a'b(1-t) = \overline{b}a'(1-t) = 0$. Hence there exists an element

$d \in R$ such that $bd = 0$ and $a(1-t)(1-d) = 0$. If $d \notin M$, then $bd = 0 \in aR$. If $1 - d \notin M$, then $(1-t)(1-d) \notin M$ and $a(1-t)(1-d) = 0 \in bR$. Thus the proof is finished. $\square$

Below we present some generalization of the definition of right Prüfer rings.

Definition 7.3.2. ([84]) A semiprime Goldie ring R is called a *right semi-Prüfer ring* if every finitely generated right R-ideal I satisfies the equalities (7.1.1)

We have the following:

Proposition 7.3.3. *Let R be a duo semiprime Goldie ring. Then the following conditions are equivalent:*

(1) R is semihereditary.

(2) R has weak dimension less than or equal to one.

(3) R is distributive.

(4) R is Gaussian.

(5) R is right semi-Prüfer ring.

Proof. The same arguments as in the proof of Theorem 7.2.4 give us (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4).

Let us assume that R is Gaussian. Then R being duo semiprime is reduced. Hence by Theorem 7.3.1 the ring R is distributive. By [80, 3.8], $R = R_1 \times \cdots \times R_n$, where $n \in \mathbb{N}$ and every R_i is right Ore domain. Since R is duo ring, every R_i is duo as well. Thus by Corollary 7.2.8 and Corollary 2.2.3, R_i is right Prüfer ring for every $i \in \{1, \ldots, n\}$. Now, [84, Theorem 24] implies that R is right semi-Prüfer ring.

Now [84, Proposition 5] completes the proof. $\square$

Remark 7.3.4. Notice that if we assume that R is duo reduced ring, then the conditions from Proposition 7.3.3 do not have to be still equivalent. Indeed, the ring constructed in Example 3.6.7 is commutative and reduced, satisfies conditions $(2), (3), (4)$ of Proposition 7.3.3 but is not semihereditary.

Regarding Question 7.2.9 it is obvious that it should be posed the following:

Question 7.3.5. *Is it true that if R is a semiprime Goldie ring which is right Gaussian, then R is right semi-Prüfer ring?*

Remark 7.3.6. [1, Example 1.15] says that if a ring R is right Bézout, then R is right Prüfer. Thus, if one would conjecture that the answer regarding Question 7.2.9 or Question 7.3.5 is negative, and would like to construct appropriete examples, then unfortunately, by Theorem 4.3.1, skew generalized power series rings with exponents from nontrivial positively strictly ordered monoid, can not provide a tool to construct such examples.

Chapter 8

Armendariz rings

This Chapter is based on:

• G. Marks, R. Mazurek, M. Ziembowski, *A unified approach to various generalizations of Armendariz rings*, Bull. Austral. Math. Soc. 81 (2010), 361-397.

As it was said below of Theorem 2.2.6, in the present chapter we will consider in detail Armendariz rings and their generalizations.

In 1974 E. P. Armendariz noted in [4] that whenever the product of two polynomials over a reduced ring R is zero, then the products of their coefficients are all zero, that is, in the polynomial ring $R[x]$ the following holds:

$$(\star)\ \text{ for any } f(x) = \sum a_i x^i,\ g(x) = \sum b_j x^j \in R[x],$$

$$\text{if } f(x)g(x) = 0, \text{ then } a_i b_j = 0 \text{ for all } i, j.$$

Nowadays the property $(\star)$ is known as the *Armendariz condition*, and rings R that satisfy $(\star)$ are called *Armendariz rings*.

The pioneering paper [67] further proposes the study of rings with an analogue of the Armendariz condition defined with respect to power series rings. This proposal has been put into effect and extended: rings satisfying an Armendariz condition for generalized power series extensions, as well as rings satisfying such a condition for monoid rings, have been studied. The condition has also been defined and investigated for skew polynomial rings, for skew power series rings, and for skew monoid rings (see the beginning of Section 8.1 for details). But although these new classes of rings were all defined using generalizations or analogues of the Armendariz condition, the theory of each class was developed separately, which led to many papers with parallel results.

In this chapter we propose a unified approach to all the above-mentioned classes of rings. The idea is to study the Armendariz condition defined for the skew generalized power series ring $R[[S,\omega]]$, where R is a ring, S is a strictly ordered monoid, and $\omega\colon S \to \mathrm{End}(R)$ is a monoid homomorphism. Since (skew) polynomial rings, (skew) power series rings, (skew) monoid rings, and generalized power series rings are particular cases of the $R[[S,\omega]]$ ring construction, the class of Armendariz rings as well as all the mentioned above classes of Armendariz-like rings are subclasses of the new class of (S,ω)-*Armendariz rings*. Hence any result on (S,ω)-Armendariz rings has its counterpart in each of the subclasses, and these counterparts follow immediately from a single proof.

In this chapter we extend to (S,ω)-Armendariz rings many results known earlier for particular types of the Armendariz-like rings. Nevertheless, we would like to underscore that some of our results are new even for Armendariz rings; for example, we prove that left chain rings are Armendariz (Corollary 8.5.3).

8.1 (S,ω)-Armendariz rings

In the present section we introduce (S,ω)-Armendariz rings.

Following Hong, Kim, and Kwak [30], we say that a ring R with an endomorphism σ is σ-*skew Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x;\sigma]$ satisfy $f(x)g(x) = 0$ then $a_i\sigma^i(b_j) = 0$ for all i,j. A stronger condition than *Armendariz* was studied by Kim, K. H. Lee, and Y. Lee in [36]. A ring R is said to be *power-serieswise Armendariz* if whenever power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x]]$ satisfy $f(x)g(x) = 0$ then $a_i b_j = 0$ for all i, j. In [45], Z. Liu extended the Armendariz notion to monoid rings. If R is a ring and S is a monoid, then R is called an *Armendariz ring relative to* S if whenever elements $f = a_1 s_1 + a_2 s_2 + \cdots + a_m s_m$ and $g = b_1 t_1 + b_2 t_2 + \cdots + b_n t_n$ of the monoid ring $R[S]$ satisfy $fg = 0$, then $a_i b_j = 0$ for all i, j. In the case of commutative monoids, Liu generalized this definition in [44] to (untwisted) generalized power series rings as follows. If R is a ring and $(S, \leqslant)$ is a commutative strictly ordered monoid, then R is called S*-Armendariz* if whenever generalized power series $f, g \in R[[S, 1]]$ satisfy $fg = 0$, we have $f(s)g(t) = 0$ for all $s, t \in S$.

We unify the above versions of Armendariz rings by introducing the following definition.

Definition 8.1.1. Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, and $\omega \colon S \to \mathrm{End}(R)$ a monoid homomorphism. We say that R is (S, ω)*-Armendariz* (or $(S, \omega, \leqslant)$*-Armendariz* to indicate the order $\leqslant$) if whenever $fg = 0$ for $f, g \in R[[S, \omega]]$, we have $f(s) \cdot \omega_s(g(t)) = 0$ for all $s, t \in S$.

If $S = \{1\}$ then every ring is (S, ω)-Armendariz. In some of our results we will stipulate that $S \neq \{1\}$ to avoid trivialities.

Example 8.1.2. Here are some special cases of (S, ω)-Armendariz rings.

(a) Suppose R is Armendariz, as in [67]. This is the special case where $S = \mathbb{N} \cup \{0\}$ under addition, with the trivial order, and ω is trivial.

(b) Suppose R is σ-skew Armendariz for some $\sigma \in \mathrm{End}(R)$, as in [30]. This is the special case where $S = \mathbb{N} \cup \{0\}$ under addition, with the trivial order, and ω is determined by $\omega(1) = \sigma$.

(c) Suppose R is power-serieswise Armendariz, as in [36]. This is the special case where $S = \mathbb{N} \cup \{0\}$ under addition, with its natural linear order, and ω is trivial.

(d) Suppose R is Armendariz relative to a monoid S, as in [45]. This is the special case where S is given the trivial order, and ω is trivial.

(e) Suppose R is S-Armendariz for some commutative, strictly ordered monoid $(S, \leqslant)$, as in [44]. This is the special case where ω is trivial (and S satisfies the extra conditions just described). □

We recall the definition of a *compatible* endomorphism from [3, Definition 2.1]:

Definition 8.1.3. An endomorphism σ of a ring R is called *compatible* if for all $a, b \in R$ we have

$$ab = 0 \Leftrightarrow a\sigma(b) = 0.$$

Compatibility arises naturally in the study of (S, ω)-Armendariz rings. To see why, suppose R is a ring and σ is an endomorphism of R. Then the skew power series ring $R[[x; \sigma]]$ is a skew generalized power series ring for $S = \mathbb{N} \cup \{0\}$ with natural order $\leqslant$ and $\omega(n) = \sigma^n$. Notice that for elements a and b of an (S, ω)-Armendariz ring R, if $ab = 0$, then $a\sigma(b) = 0$ (i.e. "half" of the definition of compatibility must hold). Indeed, define $f, g \in R[[x; \sigma]]$ as follows:

$$f = a - ax, \quad g = b + \sigma(b)x + \sigma^2(b)x^2 + \sigma^3(b)x^3 + \cdots .$$

Then $fg = 0$, and invoking the (S, ω)-Armendariz condition for the constant coefficient of f and the x-coefficient of g yields $a\sigma(b) = 0$.

Definition 8.1.4. Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. We say that R is *S-compatible* (resp. *S-rigid*) if ω_s is compatible (resp. rigid) for every $s \in S$. To indicate the homomorphism ω, we will sometimes say that R is *(S, ω)-compatible* (resp. *(S, ω)-rigid*), (the definition of a rigid endomorphism of a ring R was recalled in Section 3.5 of the Chapter 3).

Basic properties of rigid and compatible endomorphisms, proved by E. Hashemi and A. Moussavi in [25, Lemmas 2.1 and 2.2], are summarized here:

Lemma 8.1.5. *Let σ be an endomorphism of a ring R. Then:*

(i) *If σ is compatible, then σ is injective.*

(ii) *σ is compatible if and only if for all $a, b \in R$, $\sigma(a)b = 0 \Leftrightarrow ab = 0$.*

(iii) *The following conditions are equivalent:*

(1) *σ is rigid.*

(2) *σ is compatible and R is reduced.*

(3) *For every $a \in R$, $\sigma(a)a = 0 \Rightarrow a = 0$.*

It will be useful to establish criteria for the transfer of the (S, ω)-Armendariz condition from one ring to another. Let R_1 and R_2 be rings, $(S_1, \leqslant_1)$ and $(S_2, \leqslant_2)$ strictly ordered monoids, and let υ: $S_1 \to \mathrm{End}R_1$ and ω: $S_2 \to \mathrm{End}R_2$ be monoid homomorphisms. Let α: $S_1 \to S_2$ be a monoid monomorphism such that for any artinian and narrow subset T of S_1, $\alpha(T)$ is an artinian and narrow subset of S_2, and let φ: $R_1 \to R_2$ be a ring homomorphism such that for every $s \in S_1$ the following diagram is commutative:

$$\begin{array}{ccc} R_1 & \stackrel{\varphi}{\to} & R_2 \\ \upsilon_s \downarrow & & \downarrow \omega_{\alpha(s)} \\ R_1 & \stackrel{\varphi}{\to} & R_2 \end{array}$$

For $f \in R_1[[S_1, \upsilon, \leqslant_1]]$, let $\bar{f}$: $S_2 \to R_2$ be the map defined as follows:

$$\bar{f}(x) = \begin{cases} \varphi \circ f \circ \alpha^{-1}(x) & \text{if } x \in \alpha(S_1) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathrm{supp}(\bar{f}) \subseteq \alpha(\mathrm{supp}(f))$, and thus $\bar{f} \in R_2[[S_2, \omega, \leqslant_2]]$. Putting $\Phi(f) = \bar{f}$, we define a map Φ: $R_1[[S_1, \upsilon, \leqslant_1]] \to R_2[[S_2, \omega, \leqslant_2]]$. Fixing all of this notation, we have the following two lemmas, the proofs of which we suppress.

Lemma 8.1.6. *The map* $\Phi\colon R_1[[S_1, \upsilon, \leqslant_1]] \to R_2[[S_2, \omega, \leqslant_2]]$ *is a ring homomorphism, and* $\ker\Phi = (\ker\varphi)[[S_1, \upsilon]]$.

Lemma 8.1.7. *If* $\varphi\colon R_1 \to R_2$ *is injective and* R_2 *is* (S_2, ω)*-Armendariz, then* R_1 *is* (S_1, υ)*-Armendariz.*

The following proposition provides us with a method of constructing (S, ω)-Armendariz rings. Recall that an ordered monoid $(S, \leqslant)$ is *left naturally ordered* if for all $s, t \in S$, $s \leqslant t \Rightarrow t \in Ss$ (cf. [71]). We say that $(S, \leqslant)$ is *quasitotally ordered* (and that $\leqslant$ is a *quasitotal order* on S) if $\leqslant$ can be refined to an order $\preccurlyeq$ with respect to which S is a strictly, totally ordered monoid.

Proposition 8.1.8. *Let* R *be a domain,* $(S, \leqslant)$ *a strictly ordered monoid, and* $\omega\colon S \to \operatorname{End}(R)$ *a monoid homomorphism. Assume that the order* $\leqslant$ *can be refined to a strict total order* $\preccurlyeq$ *such that the monoid* $(S, \preccurlyeq)$ *is left naturally ordered. Then* R *is an* (S, ω)*-Armendariz ring.*

Proof. By Lemma 8.1.7 it suffices to show that R is $(S, \omega, \preccurlyeq)$-Armendariz. Assume that R is not an $(S, \omega, \preccurlyeq)$-Armendariz ring. Then there exist $f, g \in R[[S, \omega]]$ such that $fg = 0$ but the set $H = \{(s, t) \in S \times S \;:\; f(s) \cdot \omega_s(g(t)) \neq 0\}$ is nonempty. The sets $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ are well-ordered with respect to $\preccurlyeq$ and $H \subseteq \operatorname{supp}(f) \times \operatorname{supp}(g)$, so we can choose an element $(s_0, t_0) \in H$ minimal with respect to the lexicographic order $\preccurlyeq_{lex}$.

Suppose that there exists $(s, t) \in H \setminus \{(s_0, t_0)\}$ such that $st = s_0t_0$. By the choice of (s_0, t_0) we have $s_0 \preccurlyeq s$. Since the order $\preccurlyeq$ is strict and total, and $st = s_0t_0$, and $(s, t) \neq (s_0, t_0)$, it follows that $s_0 \prec s$. Thus $t \prec t_0$, and consequently $(s_0, t) \prec_{lex} (s_0, t_0)$. Hence the minimality of (s_0, t_0) implies that $f(s_0) \cdot \omega_{s_0}(g(t)) = 0$, and since R is a domain, we obtain $\omega_{s_0}(g(t)) = 0$. Furthermore, since $s_0 \prec s$, there exists $z \in S$ such that $s = zs_0$, and thus $\omega_s(g(t)) = \omega_z(\omega_{s_0}(g(t))) = 0$, contradicting $(s, t) \in H$.

By the above, the only element $(s, t) \in H$ with $st = s_0t_0$ is $(s, t) = (s_0, t_0)$. Therefore, since $(s_0, t_0) \in H$ and $fg = 0$, we obtain $0 \neq f(s_0) \cdot \omega_{s_0}(g(t_0)) =$

$(fg)(s_0t_0) = 0$, a contradiction. $\square$

Since the trivial order on the additive monoid $S = \mathbb{N} \cup \{0\}$ can be refined to the usual order $\leqslant$ and S is naturally ordered by $\leqslant$, from Proposition 8.1.8 and Example 8.1.2(ii), we obtain the following result of Hong, Kim, and Kwak.

Corollary 8.1.9 ([30, Proposition 10]). *If R is a domain, then R is σ-skew Armendariz for any endomorphism σ of R.*

8.2 Characterizations of (S, ω)-Armendariz rings via annihilators

In this section we will present a characterization theorem for (S, ω)-Armendariz rings in terms of one-sided annihilators. If $(S, \leqslant)$ is a strictly ordered monoid, R is a ring, $\omega \colon S \to \mathrm{End}(R)$ is a monoid homomorphism and $A = R[[S, \omega]]$, then for $\emptyset \neq X \subseteq R$,

$$X[[S, \omega]] = \{f \in A \; : \; f(s) \in X \cup \{0\} \text{ for every } s \in S\},$$

and for $\emptyset \neq Y \subseteq A$,

$$\mathrm{C}_Y = \{g(t) \; : \; g \in Y, \, t \in S\}.$$

Note that

$$X \cup \{0\} = \mathrm{C}_{X[[S,\omega]]} \quad \text{for any nonempty subset } X \text{ of } R. \tag{8.2.1}$$

Lemma 8.2.1. *Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, $\omega \colon S \to \mathrm{End}(R)$ a monoid homomorphism, and $A = R[[S, \omega]]$. If R is S-compatible, then:*

(i) *For any nonempty subset $X \subseteq R$, $r_R(X)[[S, \omega]] = r_A(X[[S, \omega]])$.*

(ii) *For any nonempty subset $X \subseteq R$, $r_R(X)$ is an ideal of R if and only if $r_R(X)[[S, \omega]]$ is an ideal of A.*

(i') *The analogue of* (i) *for left annihilators.*

(ii') *The analogue of* (ii) *for left annihilators.*

Proof. (i) The inclusion $r_R(X)[[S,\omega]] \subseteq r_A(X[[S,\omega]])$ is clear from S-compatibility. To prove the opposite inclusion, consider any $f \in r_A(X[[S,\omega]])$, $s \in S$, and $x \in X$. Since $\mathsf{c}_x \in X[[S,\omega]]$, we have $\mathsf{c}_x f = 0$ and thus $xf(s) = (\mathsf{c}_x f)(s) = 0$, which shows that $f \in r_R(X)[[S,\omega]]$.

(ii) Assume $r_R(X)$ is an ideal of R. By (i), $r_R(X)[[S,\omega]]$ is a right ideal of A. Choose any $f \in A$ and $g \in r_R(X)[[S,\omega]]$. Let $s,t \in S$. Then $g(t) \in r_R(X)$, and the S-compatibility of R yields $\omega_s(g(t)) \in r_R(X)$. By hypothesis $r_R(X)$ is an ideal of R. Thus, $f(s) \cdot \omega_s(g(t)) \in r_R(X)$. Hence for any $z \in S$ we have $(fg)(z) \in r_R(X)$, which shows that $fg \in r_R(X)[[S,\omega]]$.

Conversely, assume $r_R(X)[[S,\omega]]$ is an ideal of A. Then for any $a \in R$ and $r \in r_R(X)$ we have $\mathsf{c}_{ar} = \mathsf{c}_a\mathsf{c}_r \in A \cdot r_R(X)[[S,\omega]] \subseteq r_R(X)[[S,\omega]]$, and thus $ar = \mathsf{c}_{ar}(1) \in r_R(X)$. Hence $r_R(X)$ is an ideal of R.

(i')–(ii') The proofs are analogous. $\square$

We are now ready to characterize (S,ω)-Armendariz rings among S-compatible rings as those for which there exists a specific bijection between the sets of right (equivalently, left) annihilator ideals of R and of $R[[S,\omega]]$, generalizing annihilator characterizations of various classes of Armendariz-like rings given in [28, Proposition 3.1], [28, Proposition 3.4], and [36, Proposition 2.6].

To state the result we introduce the following notation. For a ring R we put

$$r_R(Subs(R)) = \{r_R(X) \ : \ X \text{ is a nonempty subset of } R\},$$

$$l_R(Subs(R)) = \{l_R(X) \ : \ X \text{ is a nonempty subset of } R\}.$$

Theorem 8.2.2. *Let R be a ring, $(S,\cdot,\leqslant)$ a strictly ordered monoid, $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism, and let $A = R[[S,\omega]]$. If R is S-compatible, then the following are equivalent:*

(1) R is (S,ω)-Armendariz.

(2) For any nonempty subset Y of A, $r_A(Y) = r_R(C_Y)[[S,\omega]]$.

(2′) For any nonempty subset Y of A, $l_A(Y) = l_R(C_Y)[[S,\omega]]$.

(3) The map $\varphi : r_R(Subs(R)) \to r_A(Subs(A))$; $r_R(X) \mapsto r_R(X)[[S,\omega]]$, is bijective.

(3′) The map $\varphi' : l_R(Subs(R)) \to l_A(Subs(A))$; $l_R(X) \mapsto l_R(X)[[S,\omega]]$, is bijective.

(4) The map $\psi : r_A(Subs(A)) \to r_R(Subs(R))$; $r_A(Y) \mapsto r_R(C_Y)$, is bijective.

(4′) The map $\psi' : l_A(Subs(A)) \to l_R(Subs(R))$; $l_A(Y) \mapsto l_R(C_Y)$, is bijective.

Proof. (1) $\Rightarrow$ (2) Using the S-compatibility of R, it is easy to see that $r_A(Y) \supseteq r_R(C_Y)[[S,\omega]]$. The opposite inclusion follows directly from the condition (1) and the S-compatibility of R.

(2) $\Rightarrow$ (3) By Lemma 8.2.1 $r_R(X)[[S,\omega]] = r_A(X[[S,\omega]])$ for any nonempty subset X of R. Thus $\operatorname{im}\varphi \subseteq r_A(Subs(A))$.

Clearly φ is injective. Since φ is surjective by the condition (2), φ is bijective.

(3) $\Rightarrow$ (4) From the equations (8.2.1) and Lemma 8.2.1 it follows that the composition $\psi \circ \varphi$ is the identity map of $Subs(R)$. Hence if φ is bijective, so is ψ.

(4) $\Rightarrow$ (1) Let $f, g \in A$ be such that $fg = 0$. Since by the equation (8.2.1) we have $r_R(C_{\{f\}}) = r_R(C_{C_{\{f\}}[[S,\omega]]})$, it follows that $\psi(r_A(f)) = \psi(r_A(C_{\{f\}}[[S,\omega]]))$. Therefore, (4) implies that $r_A(f) = r_A(C_{\{f\}}[[S,\omega]])$, and thus $g \in r_A(C_{\{f\}}[[S,\omega]])$. If $s \in S$, then $c_{f(s)} \in C_{\{f\}}[[S,\omega]]$, and thus $c_{f(s)}g = 0$. Hence for any $t \in S$ we have $f(s)g(t) = (c_{f(s)}g)(t) = 0$, and thus $f(s)\omega_s(g(t)) = 0$ by the S-compatibility of R, proving that R is (S,ω)-Armendariz.

The implications (1) $\Rightarrow$ (2′), (2′) $\Rightarrow$ (3′), (3′) $\Rightarrow$ (4′), and (4′) $\Rightarrow$ (1) follow by similar arguments. $\square$

As a consequence of Theorem 8.2.2, we obtain the following generalization of [28, Corollary 3.3], which was provided to counterpoint J. W. Kerr's example of a polynomial ring over a Goldie ring that is not a Goldie ring.

Corollary 8.2.3. *Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, and $\omega \colon S \to \mathrm{End}(R)$ a monoid homomorphism. Suppose that R is S-compatible and (S,ω)-Armendariz. Then R satisfies the ascending chain condition on right annihilators if and only if $R[[S,\omega]]$ satisfies the ascending chain condition on right annihilators.*

8.3 The (S,ω)-Armendariz condition and generalizations of commutativity

In this section we will obtain criteria for skew generalized power series rings to satisfy various conditions on noncommutative rings that generalize commutativity. A ring R is called *reversible* if for all $a, b \in R$ we have $ab = 0 \Leftrightarrow ba = 0$. A ring is called *2-primal* if its prime radical contains every nilpotent element of the ring. There is a substantial literature on these conditions, a survey of some of which can be found in [50]. The conditions have the following relationships, where the bottom left condition is defined with respect to any nontrivial strictly ordered monoid $(S, \leqslant)$:

$$\begin{array}{ccccc}
\text{reduced} & \Rightarrow & \text{power-serieswise Armendariz} & & \\
\Downarrow & & \Downarrow & & \\
\text{reversible} & \Rightarrow & \text{semicommutative} & \Rightarrow & \text{2-primal} \\
 & & \Downarrow & & \\
(S,\omega)\text{-Armendariz} & \Rightarrow & \text{abelian} & &
\end{array}$$

The implication "reduced $\Rightarrow$ power-serieswise Armendariz," originally established in [36, Lemma 2.3(1)], is generalized in Theorem 8.3.12. The implication "(S,ω)-Armendariz $\Rightarrow$ abelian" follows from Proposition 8.3.9(ii) below. For "power-serieswise Armendariz $\Rightarrow$ semicommutative," please see [36, Lemma 2.3(2)]. The remaining implications are well known (cf. [50] and sources cited).

In the above diagram, the six conditions *reduced*, *power-serieswise Armendariz*, *reversible*, *semicommutative*, *2-primal*, and *abelian* are equivalent for von Neumann regular rings. Thus, the characterizations of these conditions in skew generalized power series rings given below might be compared with the criteria obtained in [57] for a skew generalized power series ring to be von Neumann regular.

We first examine semicommutativity of skew generalized power series rings. Directly from Lemma 8.2.1 and Theorem 8.2.2 we obtain the following:

Theorem 8.3.1. *Let R be a ring, $(S,\leqslant)$ a strictly ordered monoid, and $\omega\colon S \to \operatorname{End}(R)$ a monoid homomorphism. Assume that R is S-compatible and (S,ω)-Armendariz. Then $R[[S,\omega]]$ is semicommutative if and only if R is.*

Combining Theorem 8.3.1 and Example 8.1.2(a) we get [67, Proposition 4.6].

In order to obtain criteria for R and $R[[S,\omega]]$ to be semicommutative, we first derive some necessary conditions for a ring to be (S,ω)-Armendariz. Recall that a monoid S is *aperiodic* if for any $s \in S\setminus\{1\}$ and $m,n\in\mathbb{N}$ with $m\neq n$ we have $s^m \neq s^n$.

Lemma 8.3.2. *Let R be a ring, $(S,\leqslant)$ a strictly ordered monoid, and $\omega\colon S \to \operatorname{End}(R)$ a monoid homomorphism. If R is (S,ω)-Armendariz, then*

(i) *S is cancellative.*

(ii) *S is aperiodic.*

(iii) *Let $x \in R$ and $s \in S\setminus\{1\}$ be such that $x\cdot\omega_s(x)\cdot\omega_{s^2}(x)\cdots\omega_{s^n}(x) = 0$ for some $n\in\mathbb{N}$. Then for all $a,b\in R$, if $ab=0$, then $a\cdot x\cdot\omega_s(b)=0$.*

Proof. (i) Let $s, t \in S$ be such that $s \neq t$. Suppose that there exists $z \in S$ such that $sz = tz$. Then in $R[[S,\omega]]$ we have $(\mathsf{e}_s - \mathsf{e}_t)\mathsf{e}_z = 0$, and since R is (S,ω)-Armendariz, it follows that $0 = (\mathsf{e}_s - \mathsf{e}_t)(s) \cdot \omega_s(\mathsf{e}_z(z)) = 1 \cdot \omega_s(1) = 1$, a contradiction. Similarly one can show that $s \neq t$ implies $zs \neq zt$.

(ii) Suppose that S is not aperiodic. Applying (i), we deduce that there exists $s \in S \setminus \{1\}$ such that $s^n = 1$ for some $n \in \mathbb{N}$. We can assume that $s^i \neq 1$ for each $i \in \{1, 2, \ldots, n-1\}$. Since $(1 - \mathsf{e}_s)(1 + \mathsf{e}_s + \mathsf{e}_{s^2} + \cdots + \mathsf{e}_{s^{n-1}}) = 0$ and R is (S,ω)-Armendariz, we obtain

$$0 = (1 - \mathsf{e}_s)(1) \cdot (1 + \mathsf{e}_s + \mathsf{e}_{s^2} + \cdots + \mathsf{e}_{s^{n-1}})(1) = 1 \cdot 1 = 1,$$

a contradiction.

(iii) Set $f = \mathsf{c}_x \mathsf{e}_s \in R[[S,\omega]]$. Since $x \cdot \omega_s(x) \cdot \omega_{s^2}(x) \cdots \omega_{s^n}(x) = 0$, it follows that $f^{n+1} = 0$, and thus

$$\mathsf{c}_a(1 - f)(1 + f + f^2 + \cdots + f^n)\mathsf{c}_b = \mathsf{c}_a\mathsf{c}_b = 0.$$

Since R is (S,ω)-Armendariz, and S is aperiodic by (ii), we obtain

$$0 = [\mathsf{c}_a(1 - f)](1) \cdot [(1 + f + f^2 + \cdots + f^n)\mathsf{c}_b](s) = a \cdot x \cdot \omega_s(b). \qquad \square$$

Proposition 8.3.3. *Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, and $\omega \colon S \to \operatorname{End}(R)$ a monoid homomorphism. If R is S-compatible and (S,ω)-Armendariz, and there exists $s \in S \setminus \{1\}$ such that $s^m \leqslant s^n$ for some positive integers $m < n$, then R and $R[[S,\omega]]$ are semicommutative.*

Proof. By Theorem 8.3.1, to prove the result it suffices to show that R is semicommutative. By [57, Lemma 4], the set $\{1, s, s^2, \ldots\}$ is artinian and narrow, and by Lemma 8.3.2(ii), for all $i, j \in \mathbb{N} \cup \{0\}$ with $i \neq j$ we have $s^i \neq s^j$. Take any $r \in R$ and define $f \in R[[S,\omega]]$ by setting

$$f(1) = 1, \qquad f(s^n) = r \cdot \omega_s(r) \cdot \omega_{s^2}(r) \cdots \omega_{s^{n-1}}(r) \quad \text{for every } n \in \mathbb{N},$$

and $f(x) = 0$ for every $x \in S \setminus \{1, s, s^2, \ldots\}$. It is easy to see that $(1 - \mathsf{c}_r\mathsf{e}_s)f = 1$. Therefore, for any $a, b \in R$ with $ab = 0$ we have $\mathsf{c}_a(1 - \mathsf{c}_r\mathsf{e}_s)f\mathsf{c}_b = 0$, and since R is (S, ω)-Armendariz, it follows that

$$[\mathsf{c}_a(1 - \mathsf{c}_r\mathsf{e}_s)](s) \cdot \omega_s((f\mathsf{c}_b)(1)) = 0.$$

Hence $-a \cdot r \cdot \omega_s(b) = 0$, and the S-compatibility of R implies that $arb = 0$. $\square$

Combining Proposition 8.3.3 and Example 8.1.2(c) we get [36, Lemma 2.3(2)].

In the proof of Theorem 8.3.12 we will need the following observation on semicommutative skew generalized power series rings.

Lemma 8.3.4. *Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, and $\omega \colon S \to \operatorname{End}(R)$ a monoid homomorphism.*

(i) *If $R[[S, \omega]]$ is semicommutative, then $ab = 0$ implies $a\omega_s(b) = 0$ for all $a, b \in R$ and all $s \in S$.*

(ii) *If $R[[S, \omega]]$ is reversible, or if $R[[S, \omega]]$ is semicommutative and S is a group, then R is S-compatible.*

Proof. Suppose $R[[S, \omega]]$ is semicommutative. Given $a, b \in R$ and $s \in S$ such that $ab = 0$, semicommutativity implies $\mathsf{c}_a\mathsf{e}_s\mathsf{c}_b = 0$, so $a\omega_s(b) = 0$. This proves (i). If, in addition, s has an inverse in S, then $a\omega_s(b) = 0$ implies $ab = 0$. Likewise, in the case where $R[[S, \omega]]$ is reversible,

$$a\omega_s(b) = 0 \Rightarrow \mathsf{c}_a\mathsf{e}_s\mathsf{c}_b = 0 \Rightarrow \mathsf{c}_b\mathsf{c}_a\mathsf{e}_s = 0 \Rightarrow ba = 0 \Rightarrow ab = 0.$$

This proves (ii). $\square$

Perhaps the greatest unsolved problem in noncommutative ring theory today is the Köthe conjecture, which posits that a ring with no nonzero nil ideals has no nonzero nil one-sided ideals. (See [65] for a discussion of the Köthe conjecture and various related problems.) The Köthe conjecture has been resolved in several special cases, including for rings with Krull dimension, for PI rings, and for algebras over uncountable fields. We will presently add S-compatible (S,ω)-Armendariz rings to this list.

For a ring R, let $\mathfrak{N}(R)$ denote the set of nilpotent elements of R, $\mathrm{N}_0(R)$ the Wedderburn radical of R (i.e. the sum of all nilpotent ideals of R), $\mathrm{Nil}_*(R)$ the prime radical of R, $\mathrm{Nil}^*(R)$ the upper nilradical of R (i.e. the largest nil ideal of R), and $\mathrm{A}(R)$ the sum of all nil left ideals of R (which coincides with the sum of all nil right ideals of R). The Köthe conjecture is equivalent to the statement that $A(R)$ is always nil, i.e. $\mathrm{Nil}^*(R) = A(R)$ for every ring R.

Proposition 8.3.5. *Let R be a ring, $(S,\leqslant)$ a strictly ordered monoid, and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. If R is (S,ω)-Armendariz and ω_s is compatible for some $s \in S \setminus \{1\}$, then*

(i) *For all $a, b \in R$ and $x \in \mathfrak{N}(R)$, $ab = 0$ implies $axb = 0$.*

(ii) *$\mathfrak{N}(R)$ is a (nonunital) subring of R.*

(iii) *$\mathrm{N}_0(R) = \mathrm{Nil}_*(R) = \mathrm{Nil}^*(R) = \mathrm{A}(R)$. In particular, the Köthe problem has a positive solution in the class of S-compatible (S,ω)-Armendariz rings.*

Proof. (i) Since ω_s is compatible for some $s \in S \setminus \{1\}$, (i) follows directly from Lemma 8.3.2(iii).

(ii) Let $x, y \in \mathfrak{N}(R)$. Then $x^n = y^n = 0$ for some $n \in \mathbb{N}$. Hence $(xy)^n = 0$ by (i), and thus $xy \in \mathfrak{N}(R)$. Clearly $(x+y)^{2n-1}$ is a finite sum of elements of the form $x^{\alpha_1}y^{\beta_1}x^{\alpha_2}y^{\beta_2}\cdots x^{\alpha_k}y^{\beta_k}$ for nonnegative integers α_i and β_i satisfying $\sum_{i=1}^{k}(\alpha_i + \beta_i) = 2n-1$. Then $\sum_{i=1}^{k}\alpha_i \geqslant n$ or $\sum_{i=1}^{k}\beta_i \geqslant n$, and thus $x^{\alpha_1+\alpha_2+\cdots+\alpha_k} = 0$ or $y^{\beta_1+\beta_2+\cdots+\beta_k} = 0$, and (i) implies that $x^{\alpha_1}y^{\beta_1}x^{\alpha_2}y^{\beta_2}\ldots x^{\alpha_k}y^{\beta_k} = 0$. Hence $x + y \in \mathfrak{N}(R)$, so $\mathfrak{N}(R)$ is a subring of R.

(iii) Let $x \in \mathrm{A}(R)$. Since $\mathrm{A}(R)$ is an ideal of R, and $\mathrm{A}(R) \subseteq \mathfrak{N}(R)$ by (ii), it follows that $RxR \subseteq \mathfrak{N}(R)$. Then $x^n = 0$ for some $n \in \mathbb{N}$, and from (i) we deduce that $(RxR)^{2n-1} = 0$. Hence $x \in \mathrm{N}_0(R)$. □

Combining Proposition 8.3.5(iii) and Example 8.1.2(a) recovers [36, Lemma 2.3(5)].

As a consequence of Propositions 8.3.3 and 8.3.5(iii) we obtain the following.

Corollary 8.3.6. *Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. If R is S-compatible and (S,ω)-Armendariz, and there exists $s \in S \setminus \{1\}$ such that $s^m \leqslant s^n$ for some positive integers $m < n$, then*

$$\mathrm{N}_0(R) = \mathrm{Nil}_*(R) = \mathrm{Nil}^*(R) = \mathrm{A}(R) = \mathfrak{N}(R).$$

Combining Corollary 8.3.6 and Example 8.1.2(c) recovers [36, Lemma 2.3(6)].

A ring R with the property that for all $a_0, a_1, b_0, b_1 \in R$ if $(a_0 + a_1x)(b_0 + b_1x) = 0$ in $R[x]$ then $a_0b_1 = a_1b_0 = 0$ in R has been called *weak Armendariz* by T.-K. Lee and T.-L. Wong in [42], and *linearly Armendariz* by Camillo and P. Nielsen in [11]. Camillo and Nielsen give a compelling argument in favor of the latter nomenclature in [11, p. 608], so we will follow their usage. Here we extend this condition to skew generalized power series rings.

Definition 8.3.7. Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. We say that R is *linearly (S,ω)-Armendariz* if for all $s \in S \setminus \{1\}$ and $a_0, a_1, b_0, b_1 \in R$, whenever $(\mathsf{c}_{a_0} + \mathsf{c}_{a_1}\mathsf{e}_s)(\mathsf{c}_{b_0} + \mathsf{c}_{b_1}\mathsf{e}_s) = 0$ in $R[[S,\omega]]$, we have $a_0b_0 = a_0b_1 = a_1 \cdot \omega_s(b_0) = a_1 \cdot \omega_s(b_1) = 0$ in R.

Obviously, all (S,ω)-Armendariz rings are linearly (S,ω)-Armendariz. However, as [42, Example 3.2] shows, a linearly (S,ω)-Armendariz ring R need not be (S,ω)-Armendariz, even in the case where R is commutative and $R[[S,\omega]] = R[x]$.

Proposition 8.3.8. *Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. The following conditions are equivalent:*

(1) *R is linearly (S,ω)-Armendariz and reduced, and ω_s is injective for every $s \in S$.*

(2) *R is S-rigid and $s^2 \notin \{1, s\}$ for every $s \in S \setminus \{1\}$.*

Proof. (1) $\Rightarrow$ (2) We first show that R is S-rigid. Let $a \in R$ and $s \in S$ be such that $a\omega_s(a) = 0$. Since R is reduced, $\omega_s(a)a = 0$; thus,

$$(\mathsf{c}_{\omega_s(a)} + \mathsf{c}_{-\omega_s(a)}\mathsf{e}_s)(\mathsf{c}_a + \mathsf{c}_{\omega_s(a)}\mathsf{e}_s) = 0.$$

Since R is linearly (S,ω)-Armendariz, it follows that $\omega_s(a) \cdot \omega_s(a) = 0$. Since R is reduced, $\omega_s(a) = 0$, and thus $a = 0$ by the injectivity of ω_s.

Now we show that for any $s \in S \setminus \{1\}$ we have $s^2 \neq 1$ and $s^2 \neq s$. If $s^2 = 1$, then $(\mathsf{c}_1 + \mathsf{c}_{-1}\mathsf{e}_s)(\mathsf{c}_1 + \mathsf{c}_1\mathsf{e}_s) = 0$ leads to $1 \cdot 1 = 0$, a contradiction. If $s^2 = s$, then $(\mathsf{c}_1 + \mathsf{c}_{-1}\mathsf{e}_s)\mathsf{c}_1\mathsf{e}_s = 0$, and again we obtain $1 \cdot 1 = 0$, a contradiction.

(2) $\Rightarrow$ (1) Since R is S-rigid, Lemma 8.1.5 implies that R is reduced and ω_s is injective for every $s \in S$. To show that R is linearly (S,ω)-Armendariz, consider any $s \in S \setminus \{1\}$ and $a_0, a_1, b_0, b_1 \in R$ with $(\mathsf{c}_{a_0} + \mathsf{c}_{a_1}\mathsf{e}_s)(\mathsf{c}_{b_0} + \mathsf{c}_{b_1}\mathsf{e}_s) = 0$. Since the elements $1, s$ and s^2 are different, it follows that

$$a_0b_0 = 0, \quad a_0b_1 + a_1 \cdot \omega_s(b_0) = 0 \quad \text{and} \quad a_1 \cdot \omega_s(b_1) = 0. \tag{8.3.1}$$

Since R is reduced and $a_0b_0 = 0$, by multiplying the second equation of (8.3.1) by b_0 we obtain

$$b_0a_1 \cdot \omega_s(b_0) = 0 \quad \Rightarrow \quad b_0a_1b_0 = 0 \quad \Rightarrow \quad a_1b_0 = 0 \quad \Rightarrow \quad a_1 \cdot \omega_s(b_0) = 0,$$

using the compatibility of ω_s. Therefore R is linearly (S,ω)-Armendariz. $\square$

In [30, Proposition 20] it was proved that for any σ-skew Armendariz ring R, the ring $R[[x;\sigma]]$ is abelian if and only if σ is idempotent-stabilizing. The first part of the following result implies that the latter condition is always satisfied, and thus all σ-skew Armendariz rings are abelian. The second part of the result generalizes [35, Lemma 7], [32, Corollary 8], and [42, Lemma 3.4].

Proposition 8.3.9. *Let R be a ring, $(S,\leqslant)$ a strictly ordered monoid, and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. If R is linearly (S,ω)-Armendariz,* *then:*

(i) *For every $s \in S$, the endomorphism ω_s is idempotent-stabilizing.*

(ii) *If S is nontrivial, then R is abelian.*

Proof. (i) The statement is trivial if $s = 1$, so assume $s \in S \setminus \{1\}$. Notice that, as in the proof of [61, Lemma 4], for any idempotent $e \in R$ we have

$$(\mathsf{c}_{1-e} + \mathsf{c}_{(1-e)\omega_s(e)}\mathsf{e}_s)(\mathsf{c}_e + \mathsf{c}_{(e-1)\omega_s(e)}\mathsf{e}_s) = 0.$$

Since R is linearly (S, ω)-Armendariz, $0 = (1-e)\omega_s(e)$. As the idempotent $e \in R$ was arbitrary, $0 = e\omega_s(1-e) = e(1-\omega_s(e))$. The equations $(1-e)\omega_s(e) = 0$ and $e(1-\omega_s(e)) = 0$ yield $\omega_s(e) = e$.

(ii) As suggested by the proof of [62, Lemma 2.4], let $a, e \in R$ with e an idempotent and $s \in S \setminus \{1\}$ be given. Then

$$(\mathsf{c}_e + \mathsf{c}_{ea(e-1)}\mathsf{e}_s)(\mathsf{c}_{1-e} + \mathsf{c}_{ea(1-e)}\mathsf{e}_s) = 0,$$

and since R is linearly (S, ω)-Armendariz, it follows that $ea(1-e) = 0$. Hence $eR(1-e) = \{0\}$ for any idempotent $e \in R$, which proves that R is abelian. $\square$

This justifies the placement of "(S, ω)-Armendariz" in the chart on page 118 above. Note that the (S, ω)-Armendariz condition does not imply any of the conditions in the first two rows of that chart, in general. For instance, [11, Example 9.3] shows that a ring can be (S, ω)-Armendariz but not 2-primal.

The following proposition shows that for any (S, ω)-Armendariz ring R the set of all idempotents of $R[[S, \omega]]$ coincides with that of R. The proposition generalizes [30, Lemma 19].

Proposition 8.3.10. *Let R be a ring, $(S, \leqslant)$ a nontrivial strictly ordered monoid, and $\omega: S \to \mathrm{End}(R)$ a monoid homomorphism. Assume that R is (S, ω)-Armendariz.*

(i) *If f is an idempotent of $R[[S, \omega]]$, then $f(1)$ is an idempotent of R and $f = \mathsf{c}_{f(1)}$.*

(ii) *$R[[S,\omega]]$ is abelian.*

Proof. (i) Since $f(f-1)=0$ and R is (S,ω)-Armendariz, we obtain $f(1)\cdot(f(1)-1)=0$, and so $f(1)$ is an idempotent of R. Moreover, for any $s\in S\setminus\{1\}$ we have $0=f(1)\cdot(f-1)(s)=f(1)f(s)$. On the other hand, since $(f-1)f=0$, it follows that $(f(1)-1)\cdot f(s)=0$, and thus $f(s)=f(1)f(s)=0$. Hence $f=\mathsf{c}_{f(1)}$.

(ii) Let $f=f^2\in R[[S,\omega]]$. Then by (i), $f=\mathsf{c}_e$ for some idempotent e of R. By Proposition 8.3.9(i), ω_s is idempotent-stabilizing. Furthermore, by Proposition 8.3.9(ii), e is central in R. Now it is easy to see that $\mathsf{c}_e g=g\mathsf{c}_e$ for every $g\in R[[S,\omega]]$. □

We now turn to reduced (S,ω)-Armendariz rings. We will characterize such rings in Theorem 8.3.12 below in the case where S belongs to a subclass of the class of unique product monoids.

Recall that a monoid S is called a *unique product monoid* (or a *u.p. monoid*, or *u.p.*) if for any two nonempty finite subsets $X,Y\subseteq S$ there exist $x\in X$ and $y\in Y$ such that $xy\neq x'y'$ for every $(x',y')\in X\times Y\setminus\{(x,y)\}$; the element xy is called a *u.p. element* of $XY=\{st\ :\ s\in X,\ t\in Y\}$. Unique product monoids and groups play an important role in ring theory, for example providing a positive case in the zero-divisor problem for group rings (cf. also [7]), and their structural properties have been extensively studied (e.g. see [16] and references therein, or [63]). The class of u.p. monoids includes the right and the left totally ordered monoids and submonoids of a free group.

For our purposes, the following, more stringent conditions are needed.

Definition 8.3.11. Let $(S,\leqslant)$ be an ordered monoid. We say that $(S,\leqslant)$ is an *artinian narrow unique product monoid* (or an *a.n.u.p. monoid*, or simply *a.n.u.p.*) if for every two artinian and narrow subsets X and Y of S there exists a u.p. element in the product XY.

For any ordered monoid $(S,\leqslant)$, we have:

$$S \text{ is commutative, torsion-free, and cancellative}$$

$$\Downarrow$$

$$(S, \leqslant) \text{ is quasitotally ordered}$$

$$\Downarrow$$

$$(S, \leqslant) \text{ is a.n.u.p.}$$

$$\Downarrow$$

$$S \text{ is u.p.}$$

The converse of the bottom implication holds if $\leqslant$ is the trivial order.

In the next chapter we will give more details, examples, and interrelationships between these and other conditions on ordered monoids.

The following theorem generalizes [12, Theorem 1], [12, Corollary 2], [12, Corollary 3], [30, Proposition 3], [30, Corollary 4], [37, Corollary 3.5], [44, Lemma 2.1], [44, Lemma 3.1], [45, Corollary 1.2], [53, Theorem A] and [61, Theorem 6].

Theorem 8.3.12. *Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, and $\omega \colon S \to \mathrm{End}(R)$ a monoid homomorphism. Consider the following six conditions:*

(i) *R is reduced, and whenever $f, g \in R[[S, \omega]]$ satisfy $fg = 0$, then $f(s)g(t) = 0$ for all $s, t \in S$.*

(ii) *R is (S, ω)-Armendariz and reduced, and ω_s is injective for every $s \in S$.*

(iii) *R is linearly (S, ω)-Armendariz and reduced, and ω_s is injective for every $s \in S$.*

(iv) *$R[[S, \omega]]$ is reduced.*

(v) *R is semiprime, and the ring $R[[S, \omega]]$ is reversible.*

(vi) *R is S-rigid.*

Then:

$$\begin{array}{ccccc} \text{(i)} & \Leftrightarrow & \text{(ii)} & \Rightarrow & \text{(iii)} \\ \Downarrow & & & & \Downarrow \\ \text{(iv)} & \Rightarrow & \text{(v)} & \Rightarrow & \text{(vi)} \end{array}$$

If $(S, \leqslant)$ is a.n.u.p., then all six conditions are equivalent.

Proof. First assume that $(S, \leqslant)$ is strictly ordered but not necessarily a.n.u.p.

(ii) $\Rightarrow$ (i) Assume (ii). Let $f, g \in R[[S, \omega]]$ be such that $fg = 0$, and let $s, t \in S$. We must show $f(s)g(t) = 0$. Since R is (S, ω)-Armendariz, $f(s) \cdot \omega_s(g(t)) = 0$, so the case $s = 1$ is done. Suppose $s \neq 1$, and set $r = f(s)g(t)$. Because R is semicommutative, $r\omega_s(r) = 0$. Because R is reversible, $\omega_s(r)r = 0$. Hence for $h = \mathsf{c}_{\omega_s(r)} + \mathsf{c}_{\omega_s(r)}\mathsf{e}_s$ and $k = \mathsf{c}_r - \mathsf{c}_{\omega_s(r)}\mathsf{e}_s$ in $R[[S, \omega]]$ we have $hk = 0$. The (S, ω)-Armendariz hypothesis implies $0 = h(s) \cdot \omega_s(k(1)) = \omega_s(r)^2$. Now, since R is reduced and ω_s is injective, $0 = r = f(s)g(t)$.

(ii) $\Rightarrow$ (iii) Trivial.

(iii) $\Rightarrow$ (vi) Proposition 8.3.8.

(i) $\Rightarrow$ (iv) For any $f \in R[[S, \omega]]$, if $f^2 = 0$ then (i) implies $f = 0$.

(iv) $\Rightarrow$ (v) Trivial.

(v) $\Rightarrow$ (vi) Since R is reversible and semiprime, it is reduced. Since R is S-compatible by Lemma 8.3.4(ii), it is S-rigid by Lemma 8.1.5(iii).

(i) $\Rightarrow$ (ii) It is easy to see that if R is S-compatible and (i) holds, then (ii) holds. We have already shown that (i) implies (vi). From (vi) and and Lemma 8.1.5(iii), we infer that R is S-compatible.

Now assume that $(S, \leqslant)$ is a.n.u.p.

(vi) $\Rightarrow$ (ii) By Lemma 8.1.5(iii) R is reduced and ω_s is injective for every $s \in S$.

Suppose there exist $f, g \in R[[S, \omega]]$ such that $fg = 0$ and $f(s) \cdot \omega_s(g(t)) \neq 0$ for some $s, t \in S$. Since R is reduced, the intersection of all minimal prime ideals of R is equal to (0). Hence there exists a minimal prime ideal P of R such that $f(s) \cdot \omega_s(g(t)) \notin P$, and thus the sets $X = \{x \in S : f(x) \notin P\}$ and $Y = \{y \in S : (\exists x \in S)\ \omega_x(g(y)) \notin P\}$ are nonempty. Since $X \subseteq \mathrm{supp}(f)$ and $Y \subseteq \mathrm{supp}(g)$, X and Y are artinian and narrow subsets of S, and since S is an a.n.u.p. monoid, there exists $(a, b) \in X \times Y$ such that ab is a u.p. element of XY. Since $fg = 0$, we have

$$0 = (fg)(ab) = f(a) \cdot \omega_a(g(b)) + \sum_{(u,v)\in \mathrm{X}_{ab}(f,g)\setminus\{(a,b)\}} f(u) \cdot \omega_u(g(v)). \qquad (8.3.2)$$

Observe that if $(u, v) \in \mathrm{X}_{ab}(f, g) \setminus \{(a, b)\}$, then since ab is a u.p. element of XY, we have $u \notin X$ or $v \notin Y$, and thus $f(u) \cdot \omega_u(g(v)) \in P$. Hence (8.3.2) implies that $f(a) \cdot \omega_a(g(b)) \in P$. Since each minimal prime ideal of a reduced ring is completely prime (see [39, Lemma 12.6]) and $a \in X$, it follows that $\omega_a(g(b)) \in P$. It is not hard to show that, since R is reduced, for every $r \in R$ and every $n \in \mathbb{N}$ we have $r_R(r) = l_R(r) = r_R(r^n) = l_R(r^n)$. Therefore, if $l_R(\omega_a(g(b))) \subseteq P$, then the set

$$Z = \{z_1 z_2 \cdots z_m \ : \ m \in \mathbb{N}, \ z_i = \omega_a(g(b)) \text{ or } z_i \in R \setminus P \text{ for each } i\}$$

would be a multiplicatively closed set disjoint from $\{0\}$ and properly containing $R \setminus P$, which contradicts P being a minimal prime. Therefore $l_R(\omega_a(g(b))) \nsubseteq P$, so $t \cdot \omega_a(g(b)) = 0$ for some $t \in R \setminus P$. Because R is S-compatible, we have $t \cdot \omega_x(g(b)) = 0$ for every $x \in S$, whence $b \notin Y$. This final contradiction proves that R is (S, ω)-Armendariz, establishing (ii). □

Example 8.3.13. The following counterexamples delimit Theorem 8.3.12.

(a) Let R be a field of characteristic char $R \neq 2$, let $S = \{1, s\}$ be the 2-element group (with the trivial order), and let $\omega\colon S \to \mathrm{End}(R)$ be the trivial map. Clearly $R[[S, \omega]]$ is reduced. However, the equation $(1 - \mathsf{e}_s)(1 + \mathsf{e}_s) = 0$ shows that R is not linearly (S, ω)-Armendariz. Thus, (iv) $\not\Rightarrow$ (iii) in Theorem 8.3.12 in general.

(b) Let $(S, \leqslant)$ be the ordered monoid constructed in Example 9.1.8. The monoid S admits a strict total ordering (and hence is a u.p. monoid), but $(S, \leqslant)$ is not a.n.u.p. Let R be any field of characteristic 2 and define maps $f, g\colon S \to R$

as follows:

$$f(s) = \begin{cases} 1 & \text{if } s = x_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{otherwise,} \end{cases}$$

$$g(s) = \begin{cases} 1 & \text{if } s = X_j \text{ for some } j \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Since the sets $\{x_i : i \in \mathbb{N}\}$ and $\{X_j : j \in \mathbb{N}\}$ are artinian and narrow, we have $f, g \in R[[S, 1]]$. For any $(s, t) \in \operatorname{supp}(f) \times \operatorname{supp}(g)$, there are exactly two pairs (x_i, X_j) with $x_i X_j = st$. Therefore, $fg = 0$. On the other hand one can easily verify that $gf \neq 0$; therefore, $R[[S, 1]]$ is not reversible. Since S is a u.p. monoid and R is S-rigid, Proposition 8.3.8 shows that R is linearly $(S, 1)$-Armendariz. Thus, (iii) $\not\Rightarrow$ (v) in Theorem 8.3.12 in general.

(c) Let $R = \mathbb{F}_2$, let $S = Q_8$, the quaternion group of order 8, with the trivial order, and let ω: $S \to \operatorname{End}(R)$ be the trivial map. Then $R[[S, \omega]]$ is the group algebra RS, which by [49, Example 7] is reversible but not reduced. Thus, (v) $\not\Rightarrow$ (iv) in Theorem 8.3.12 in general.

(d) Let $R = \mathbb{F}_2$, let $S = D_8$, the dihedral group of order 8, with the trivial order, and let ω: $S \to \operatorname{End}(R)$ be the trivial map. Then $R[[S, \omega]]$ is the group algebra RS, which by [49, p. 316] is not reversible. Letting $s \in S$ be any element of order 2, we have $(1 + \mathsf{e}_s)(1 + \mathsf{e}_s) = 0$. Therefore, R is not linearly (S, ω)-Armendariz. Obviously R is S-rigid. Thus, (vi) $\not\Rightarrow$ (iii) and (vi) $\not\Rightarrow$ (v) in Theorem 8.3.12 in general. □

It is of interest to consider a condition absent from Theorem 8.3.12: semicommutativity. Semicommutativity is of course a central issue for Armendariz rings (and generalizations thereof), dating back to their inception: see [67, §4]. Theorem 8.3.12 tells us that under appropriate circumstances the *reduced* condition on a skew generalized power series ring is equivalent to the generally weaker condition *reversible*. What about the still weaker condition *semicommutative*?

Theorem 8.3.1 apparently has no bearing on this problem, since the hypotheses of Theorem 8.3.1 already entail most of condition (ii) of Theorem 8.3.12. This leaves us with the following open question:

Question 8.3.14. *Suppose R is a semiprime ring, $(S, \leqslant)$ is a strictly ordered a.n.u.p. monoid, and $\omega\colon S \to \mathrm{End}(R)$ is a monoid homomorphism. If the skew generalized power series ring $R[[S,\omega]]$ is semicommutative, must $R[[S,\omega]]$ be reversible (and therefore reduced)?*

A power series ring over a 2-primal ring need not be 2-primal, as examples in [31] and [47] show. Nevertheless, under appropriate conditions, a skew generalized power series ring will be 2-primal. If R is a ring, $(S, \leqslant)$ a strictly ordered monoid, and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism, then a subset X of R is an S-*invariant* if $\omega_s(X) \subseteq X$ for all $s \in S$.

Theorem 8.3.15. *Let R be a ring, $(S, \leqslant)$ a strictly ordered a.n.u.p. monoid, and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. Suppose that $\mathrm{Nil}_*(R)$ is a nilpotent ideal, and suppose that for every $s \in S$ and every $a \in R$, if $aR\omega_s(a) \subseteq \mathrm{Nil}_*(R)$, then a is nilpotent. Then the skew generalized power series ring $R[[S,\omega]]$ is 2-primal if and only if R is 2-primal.*

Proof. If the ring $A = R[[S,\omega]]$ is 2-primal, then R is 2-primal by [6, Proposition 2.2].

Conversely, suppose R is 2-primal, and assume the prime radical $I = \mathrm{Nil}_*(R)$ satisfies $I^n = (0)$ for some $n \in \mathbb{N}$. Let $\pi\colon R \to R/I = \overline{R}$ be the canonical map. Since R is 2-primal, its prime radical is S-invariant. Therefore $J = I[[S,\omega]]$ is an ideal of A, and we have a monoid homomorphism $\upsilon\colon \mathrm{End}(R) \to \mathrm{End}(\overline{R})$ given by $\upsilon(\tau)(\overline{x}) = \overline{\tau(x)}$ for each $\overline{x} \in \overline{R}$. The surjective ring homomorphism $A \to \overline{R}[[S,\upsilon\circ\omega]]$ given by $f \mapsto \pi \circ f$ induces an isomorphism $A/J \cong \overline{R}[[S,\upsilon\circ\omega]]$.

Suppose that for $s \in S$ and $a \in R$ we have $\overline{a} \cdot (\upsilon \circ \omega)_s(\overline{a}) = \overline{0}$ in $\overline{R}$. Since R is 2-primal, $\overline{R}$ is reduced and hence semicommutative. Therefore, $\overline{a} \cdot \overline{R} \cdot (\upsilon\circ\omega)_s(\overline{a}) = \{\overline{0}\}$, whence $aR\omega_s(a) \subseteq I$. By hypothesis, then, a is nilpotent, so $\overline{a} = \overline{0}$ in $\overline{R}$. We have shown that $(\upsilon \circ \omega)_s$ is a rigid endomorphism of $\overline{R}$, for arbitrary $s \in S$.

Thus, by Theorem 8.3.12, the ring $A/J \cong \overline{R}[[S, \upsilon \circ \omega]]$ is reduced. Clearly $J^n = (0)$, so A is 2-primal. $\square$

Remark 8.3.16. In Theorem 8.3.15, if ω is trivial then the condition "$aR\omega_s(a) \subseteq \mathrm{Nil}_*(R)$ implies a nilpotent" is vacuous. So [44, Theorem 2.3] is a special case of Theorem 8.3.15.

Let R be a semiprime left Goldie ring, and let $\mathcal{C}$ denote the set of regular elements of R. If $\sigma \in \mathrm{End}(R)$ is injective, then $\sigma(\mathcal{C}) \subseteq \mathcal{C}$ by [33, Proposition 2.4]. Therefore, if $Q = Q^{\ell}_{\mathrm{cl}}(R)$ is the classical left ring of quotients of R, then one can verify that σ extends (uniquely) to an endomorphism $\widetilde{\sigma}$ of Q defined by $\widetilde{\sigma}(b^{-1}a) = \sigma(b)^{-1}\sigma(a)$ for all $a \in R$ and $b \in \mathcal{C}$.

In this setting, if S is a monoid and $\omega\colon S \to \mathrm{End}(R)$ is a monoid homomorphism such that ω_s is injective for every $s \in S$, then there is an induced monoid homomorphism $\widetilde{\omega}\colon S \to \mathrm{End}(Q)$ defined by

$$\widetilde{\omega}_s = \widetilde{(\omega_s)} \qquad \text{for each } s \in S.$$

Notice that $\widetilde{\omega}_s$ is injective for every $s \in S$.

The following result generalizes [61, Theorem 10].

Theorem 8.3.17. *Let R be a semiprime left Goldie ring, $(S, \leqslant)$ a nontrivial strictly ordered a.n.u.p. monoid, and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism such that ω_s is injective for every $s \in S$. Let $Q = Q^{\ell}_{\mathrm{cl}}(R)$ denote the classical left ring of quotients of R, and $\widetilde{\omega}\colon S \to \mathrm{End}(Q)$ the induced monoid homomorphism. Then the following conditions are equivalent:*

(1) *R is (S, ω)-Armendariz.*

(2) *R is linearly (S, ω)-Armendariz.*

(3) *R is (S, ω)-rigid.*

(4) *Q is $(S,\widetilde{\omega})$-Armendariz.*

(5) *Q is linearly $(S,\widetilde{\omega})$-Armendariz.*

(6) *Q is $(S,\widetilde{\omega})$-rigid.*

Proof. (1) $\Rightarrow$ (2) Trivial.

(2) $\Rightarrow$ (5) We have to show that for any $p_0, p_1, q_0, q_1 \in Q$ and $s \in S \setminus \{1\}$,

$$\text{if } (\mathsf{c}_{p_0}+\mathsf{c}_{p_1}\mathsf{e}_s)(\mathsf{c}_{q_0}+\mathsf{c}_{q_1}\mathsf{e}_s) = 0 \text{ in } Q[[S,\widetilde{\omega}]], \text{ then } p_0q_1 = p_1\cdot\widetilde{\omega}_s(q_0) = 0. \quad (8.3.3)$$

Now, there exist $a_0, a_1, b_0, b_1, u \in R$ such that u is regular and $p_i = u^{-1}a_i$, $q_i = u^{-1}b_i$ for $i = 1, 2$. Furthermore, for some $d_0, d_1, v \in R$ with v regular, we can write $a_0u^{-1} = v^{-1}d_0$ and $a_1\omega_s(u)^{-1} = v^{-1}d_1$. Now it is easy to see that in $R[[S,\omega]]$ we have $(\mathsf{c}_{d_0}+\mathsf{c}_{d_1}\mathsf{e}_s)(\mathsf{c}_{b_0}+\mathsf{c}_{b_1}\mathsf{e}_s) = 0$. Since R is linearly (S,ω)-Armendariz, we obtain $d_0b_1 = d_1\cdot\omega_s(b_0) = 0$. Now $p_0q_1 = p_1\cdot\widetilde{\omega}_s(q_0) = 0$ follows easily, proving (8.3.3).

(4) $\Rightarrow$ (5) Trivial.

(5) $\Rightarrow$ (6) Assuming (5), Proposition 8.3.9(ii) implies that Q is abelian. Being semisimple, Q is reduced. Hence (6) holds by Theorem 8.3.12.

(6) $\Rightarrow$ (4) Holds by Theorem 8.3.12.

(6) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) This implication follows from Theorem 8.3.12. $\square$

Applying Theorem 8.3.17 to Example 8.1.2(a), we obtain the following improvement of [35, Proposition 18] (which was, in turn, an improvement of [2, Theorem 7]), recovering [42, Theorem 3.3].

Corollary 8.3.18. *Suppose that R is a semiprime left Goldie ring. Then R is Armendariz if and only if R is reduced.*

To illustrate this corollary, let $F = k\langle x, y\rangle$ be the free algebra on two noncommuting indeterminates over a field k, and consider the two factor rings $R_1 = F/Fx^2F$

and $R_2 = F/(Fx^2F + Fy^2F)$. Both R_1 and R_2 are prime rings, and one can directly check that R_1 is Armendariz but R_2 is not. In fact, R_1 is a construction of Camillo and Nielsen in [11, Example 9.3], apparently the first example in the literature of an Armendariz ring that is not 2-primal. Camillo and Nielsen's proof that R_1 is Armendariz is based on an intricate calculation of zero-divisors. In contrast, Corollary 8.3.18 provides a "structural" proof that R_2 is not Armendariz: it is noetherian and prime but not reduced.

8.4 The (S, ω)-Armendariz condition and ring extensions

It is easy to see that if I is a *reduced ideal* of a ring R (i.e. I is an ideal of R such that $x^2 = 0 \Rightarrow x = 0$ for every $x \in I$), then for any $a, b \in R$, $ab = 0$ implies $aIb = \{0\}$. We will use this observation freely in the following proof.

Proposition 8.4.1. *Let R be a ring, $(S, \leqslant)$ a strictly totally ordered monoid and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism such that R is S-compatible. Let $f, g \in R[[S, \omega]]$ be such that for some reduced ideal I of R,*

$$(fg)(xy) = 0 \Rightarrow f(x) \cdot \omega_x(g(y)) \in I \quad \textit{for any} \quad x, y \in S.$$

Then for any $s \in S$ the following are equivalent:

(1) *$f(x) \cdot \omega_x(g(y)) = 0$ for any $x, y \in S$ such that $xy \leqslant s$.*

(2) *$(fg)(z) = 0$ for any $z \leqslant s$.*

Proof. (1) $\Rightarrow$ (2) This is obvious (and it requires no assumptions about the order $\leqslant$ or the existence of a reduced ideal I).

(2) $\Rightarrow$ (1) Suppose the implication fails. Since the sets $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are well-ordered, we can choose an element $(x_0, y_0) \in S \times S$ minimal with respect to the lexicographic order such that

$$x_0y_0 \leqslant s \quad \text{and} \quad f(x_0) \cdot \omega_{x_0}(g(y_0)) \neq 0.$$

Hence there exist $n \in \mathbb{N}$ and $(x_1, y_1), \ldots, (x_n, y_n) \in S \times S \setminus \{(x_0, y_0)\}$ such that

$$0 = (fg)(x_0 y_0) = \sum_{i=0}^{n} f(x_i) \cdot \omega_{x_i}(g(y_i)) \tag{8.4.1}$$

and for each $i \in \{0, 1, \ldots, n\}$ we have $x_i y_i = x_0 y_0$ and $f(x_i) \cdot \omega_{x_i}(g(y_i)) \neq 0$. By the choice of (x_0, y_0), for each $i \geqslant 1$ we have $x_0 < x_i$, hence $y_i < y_0$, and thus $f(x_0) \cdot \omega_{x_0}(g(y_i)) = 0$. Now the compatibility of ω_{x_0} and ω_{x_i} implies that $f(x_0) \cdot \omega_{x_i}(g(y_i)) = 0$. Hence for every $i \geqslant 1$ we have $f(x_0) \cdot I \cdot \omega_{x_i}(g(y_i)) = 0$, and since $f(x_0) \cdot \omega_{x_0}(g(y_0)) \in I$ by assumption, multiplying equation (8.4.1) on the left by $[f(x_0) \cdot \omega_{x_0}(g(y_0))]^2$ yields $0 = [f(x_0) \cdot \omega_{x_0}(g(y_0))]^3$. Since I is reduced, it follows that $f(x_0) \cdot \omega_{x_0}(g(y_0)) = 0$, a contradiction. □

As a corollary of the above result we obtain the following generalization of [45, Proposition 1.4].

Corollary 8.4.2. *Let R be a ring, $(S, \leqslant)$ a quasitotally ordered monoid and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. Assume that R is S-compatible, and that there exists a reduced ideal I of R such that for any $f, g \in R[[S, \omega]]$, if $fg = 0$, then $f(s) \cdot \omega_s(g(t)) \in I$ for all $s, t \in S$. Then R is (S, ω)-Armendariz.*

Proof. By hypothesis, the order $\leqslant$ can be refined to a strict total order $\preccurlyeq$. Since Proposition 8.4.1 implies that R is $(S, \omega, \preccurlyeq)$-Armendariz, Lemma 8.1.7 completes the proof. □

As an immediate consequence of Corollary 8.4.2, we obtain the following extension property for (S, ω)-Armendariz rings.

Corollary 8.4.3. *Let R be a ring, $(S, \leqslant)$ a quasitotally ordered monoid and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism. Suppose that R is S-compatible and that there exists an S-invariant, reduced ideal $I \subseteq R$ such that the factor ring R/I is $(S, \overline{\omega})$-Armendariz, where $\overline{\omega}\colon S \to \mathrm{End}(R/I)$ is the induced monoid homomorphism. Then R is (S, ω)-Armendariz.*

As it was said in Remark 6.1.1 Anderson and Camillo proved that for any ring R and any integer $n \geqslant 2$, the factor ring $R[x]/(x^n)$ is Armendariz if and only if R

is reduced. As we will see in Corollary 8.4.6, Anderson and Camillo's result is a consequence of the following theorem.

Theorem 8.4.4. *Let R be a ring, $(S,\leqslant)$ a strictly well-ordered monoid, $\omega\colon S\to \mathrm{End}(R)$ a monoid homomorphism. Fix $s\in S$. Then the set*

$$I_s=\{f\in R[[S,\omega]]\ :\ f(x)=0\ \text{ for every }\ x\leqslant s\}$$

is a proper ideal of $R[[S,\omega]]$. Assume that $(T,\leqslant')$ is a nontrivial quasitotally ordered monoid, and consider the following four conditions:

(i) *ω_x is rigid for every $x\in S$ satisfying $x\leqslant s$.*

(ii) *$R[[S,\omega]]/I_s$ is $(T,1,\leqslant')$-Armendariz.*

(iii) *$R[[S,\omega]]/I_s$ is $(T,1,\leqslant'')$-Armendariz where $\leqslant''$ is the trivial order.*

(iv) *$R[[S,\omega]]/I_s$ is linearly $(T,1,\leqslant'')$-Armendariz where $\leqslant''$ is the trivial order.*

In general,

$$\text{(i)}\Rightarrow\text{(ii)}\Rightarrow\text{(iii)}\Rightarrow\text{(iv)}.$$

Now assume, moreover, that $s\neq 1$ and either of the following conditions holds:

(a) *for every $x\in S\setminus\{1\}$ such that $x\leqslant s$, ω_x is injective and there exists $n\in\mathbb{N}$ such that $s<x^n$, or*

(b) *ω is trivial.*

Then

$$\text{(i)}\Leftrightarrow\text{(ii)}\Leftrightarrow\text{(iii)}\Leftrightarrow\text{(iv)}.$$

Proof. Since $(S,\leqslant)$ is a well-ordered monoid, $1\leqslant x$ for all $x\in S$, and it easily follows that I_s is a proper ideal of $R[[S,\omega]]$.

(i) $\Rightarrow$ (ii) By Lemma 8.1.7, without loss of generality we can assume that $\leqslant'$ is already a total order. Suppose $R[[S,\omega]]/I_s$ is not $(T,1,\leqslant')$-Armendariz. Pick $F,G\in(R[[S,\omega]]/I_s)[[T,1]]$ such that $FG=0$ and $F(t)G(t')\neq 0$ for some $t,t'\in$

T. For any $z \in T$ we have $F(z), G(z) \in R[[S,\omega]]/I_s$ and thus there exist $f_z, g_z \in R[[S,\omega]]$ such that $F(z) = \overline{f_z}$ and $G(z) = \overline{g_z}$ (where a bar denotes images modulo I_s). We will retain this notation for the remainder of the proof. Whenever $F(z)$ (resp. $G(z)$) is 0, we will choose $f_z = 0$ (resp. $g_z = 0$).

Define the following total order $\preccurlyeq$ on $S \times S \times T \times T$:

$$(s_1, s_2, t_1, t_2) \prec (s_3, s_4, t_3, t_4) \Leftrightarrow \begin{cases} s_1 s_2 < s_3 s_4, \text{ or} \\ s_1 s_2 = s_3 s_4 \text{ and } s_1 < s_3, \text{ or} \\ s_1 = s_3 \text{ and } s_2 = s_4 \text{ and } t_1 <' t_3, \text{ or} \\ s_1 = s_3 \text{ and } s_2 = s_4 \text{ and } t_1 = t_3 \text{ and } t_2 <' t_4. \end{cases}$$

Then $S \times S \times \mathrm{supp}(F) \times \mathrm{supp}(G)$ is well-ordered under $\preccurlyeq$, so there exists $(x, y, c, d) \in S \times S \times \mathrm{supp}(F) \times \mathrm{supp}(G)$ minimal for the property $f_c(x) \cdot \omega_x(g_d(y)) \neq 0$. Since $f_t g_{t'} \notin I_s$, we have $xy \leqslant s$.

We claim that for any element $(x', y', c', d') \in S \times S \times T \times T$,

$$\begin{aligned} & x'y' = xy \text{ and } c'd' = cd \\ & \quad \Rightarrow \quad f_{c'}(x') \cdot \omega_{x'}(g_{d'}(y')) \cdot f_c(x) = 0 \text{ or } (x, y, c, d) = (x', y', c', d'). \quad (8.4.2) \end{aligned}$$

Indeed, it suffices to consider $(x', y', c', d') \in S \times S \times \mathrm{supp}(F) \times \mathrm{supp}(G)$ for which $(x, y, c, d) \prec (x', y', c', d')$. Then either $x < x'$, or else $x = x'$ and $y = y'$ and $c <' c'$. If $x < x'$, then $xy' < xy$, hence $(x, y', c, d') \prec (x, y, c, d)$. If $x = x'$ and $y = y'$ and $c <' c'$, then $cd = c'd'$ implies $d' <' d$, and again $(x, y', c, d') \prec (x, y, c, d)$. In either case, the minimal choice of (x, y, c, d) implies $f_c(x) \cdot \omega_x(g_{d'}(y')) = 0$. Since $x \leqslant s$ and $x' \leqslant s$, (i) implies that ω_x and $\omega_{x'}$ are rigid, and $\omega_{x'}(g_{d'}(y')) \cdot f_c(x) = 0$ follows. This proves (8.4.2).

Since $FG = 0$, for some $n \in \mathbb{N} \cup \{0\}$ and $(c_1, d_1), \dots, (c_n, d_n) \in T \times T \setminus \{(c, d)\}$

such that $c_i d_i = cd$ we have

$$0 = (FG)(cd) = F(c)G(d) + \sum_{i=1}^{n} F(c_i)G(d_i).$$

Since $xy \leqslant s$,

$$f_c g_d + \sum_{i=1}^{n} f_{c_i} g_{d_i} \in I_s \qquad \Longrightarrow \qquad (f_c g_d)(xy) + \sum_{i=1}^{n} (f_{c_i} g_{d_i})(xy) = 0.$$

By (8.4.2) we have $(f_{c_i} g_{d_i})(xy) \cdot f_c(x) = 0$ for every i. Therefore,

$$0 = (f_c g_d)(xy) \cdot f_c(x) + \sum_{i=1}^{n} (f_{c_i} g_{d_i})(xy) \cdot f_c(x) = (f_c g_d)(xy) \cdot f_c(x).$$

There exist $m \in \mathbb{N} \cup \{0\}$ and $(x_1, y_1), \ldots, (x_m, y_m) \in S \times S \setminus \{(x, y)\}$ with $x_j y_j = xy$ such that

$$(f_c g_d)(xy) = f_c(x) \cdot \omega_x(g_d(y)) + \sum_{j=1}^{m} f_c(x_j) \cdot \omega_{x_j}(g_d(y_j)).$$

By (8.4.2) we have $f_c(x_j) \cdot \omega_{x_j}(g_d(y_j)) \cdot f_c(x) = 0$ for every j. Therefore, $f_c(x) \cdot \omega_x(g_d(y)) \cdot f_c(x) = 0$. Since (i) implies that R is a reduced ring, we obtain $f_c(x) \cdot \omega_x(g_d(y)) = 0$, a contradiction.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) These are obvious.

Finally, assume that $s \neq 1$ *and condition* (a) *or* (b) *holds.*

(iv) $\Rightarrow$ (i) Assume condition (a), and let $x \in S$ satisfy $x \leqslant s$. If $x = 1$ then ω_x being rigid amounts to R being reduced, which is the case because ω_s is rigid as we will show in a moment. Now suppose $x \neq 1$. Condition (a) implies that for some $n \in \mathbb{N}$ we have $x^n \leqslant s < x^{n+1}$. Assume that $r \in R$ satisfies $\omega_x(r) \cdot r = 0$. Fix $t \in T \setminus \{1\}$, and put

$$F = \overline{\mathsf{e}_x}\, 1 + \overline{\mathsf{c}_{\omega_x(r)}}\, t, \quad G = \overline{\mathsf{e}_{x^n}}\, 1 - \overline{\mathsf{c}_r \mathsf{e}_{x^{n-1}}}\, t \quad \in \quad (R[[S,\omega]]/I_s)[T].$$

Then $FG = 0$. By (iv), $\mathsf{c}_{\omega_x(r)} \mathsf{e}_{x^n} \in I_s$, which implies $\omega_x(r) = 0$; by condition (a),

$r = 0$. From Lemma 8.1.5(iii) we conclude that ω_x is rigid.

Assume condition (b). A similar argument with $F = \overline{e_s}\,1 + \overline{c_r}\,t$ and $G = \overline{e_s}\,1 - \overline{c_r}\,t$ shows that $r^2 = 0$ implies $r = 0$. □

Remark 8.4.5. Example 21 in [43] shows that the injectivity hypothesis in (a) of the second part of Theorem 8.4.4 is essential.

As a consequence of Theorem 8.4.4 we obtain the following generalization of [42, Theorem 3.1] and [43, Theorem 20].

Corollary 8.4.6. *If σ is an injective endomorphism of a ring R, and $n \geqslant 2$ is an integer, then the following conditions are equivalent:*

(1) *$R[x, \sigma]/(x^n)$ is Armendariz.*

(2) *$R[[x, \sigma]]/(x^n)$ is Armendariz.*

(3) *$R[x, \sigma]/(x^n)$ is linearly Armendariz.*

(4) *$R[[x, \sigma]]/(x^n)$ is linearly Armendariz.*

(5) *$R[x, \sigma]/(x^n)$ is power-serieswise Armendariz.*

(6) *$R[[x, \sigma]]/(x^n)$ is power-serieswise Armendariz.*

(7) *σ is rigid.*

Proof. The equivalences (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) follow from Theorem 8.4.4. The rest follows from the isomorphism $R[x, \sigma]/(x^n) \cong R[[x, \sigma]]/(x^n)$. □

Note that [2, Theorem 5] is the $\sigma = 1$ case of (1) $\Leftrightarrow$ (7) in Corollary 8.4.6.

Let R be a ring, $(S, \leqslant_S)$ and $(T, \leqslant_T)$ strictly ordered monoids, and $\omega\colon S \to \mathrm{End}(R)$ and $\upsilon\colon T \to \mathrm{End}(R)$ monoid homomorphisms such that

$$\omega_s \circ \upsilon_t = \upsilon_t \circ \omega_s \qquad \text{for all } s \in S \text{ and } t \in T.$$

It is easy to verify that the following maps are monoid homomorphisms:

- $\bar{\omega}\colon S \to \mathrm{End}(R[[T, \upsilon]])$, where $\bar{\omega}_s(g) = \omega_s \circ g$ for all $s \in S$ and $g \in R[[T, \upsilon]]$,

- $\bar{\upsilon}\colon T \to \mathrm{End}(R[[S,\omega]])$, where $\bar{\upsilon}_t(f) = \upsilon_t \circ f$ for all $t \in T$ and $f \in R[[S,\omega]]$,
- $\omega \times \upsilon\colon S \times T \to \mathrm{End}(R)$, where $(\omega \times \upsilon)_{(s,t)} = \omega_s \circ \upsilon_t$ for every $(s,t) \in S \times T$.

Proposition 8.4.7. *Let R, S, T, ω and υ be as above. Assume that the ring R is reduced, the monoids $(S, \leqslant_S)$ and $(T, \leqslant_T)$ are a.n.u.p., and for all $s \in S$ and $t \in T$ the endomorphisms ω_s and υ_t are injective. Then the monoid $S \times T$ is quasitotally ordered by the induced lexicographic order and the following conditions are equivalent:*

(1) *R is (S,ω)-Armendariz and (T,υ)-Armendariz,*

(2) *$R[[S,\omega]]$ is reduced and $(T,\bar{\upsilon})$-Armendariz,*

(3) *$R[[T,\upsilon]]$ is reduced and $(S,\bar{\omega})$-Armendariz,*

(4) *$R[[S,\omega]]$ is $(T,\bar{\upsilon})$-Armendariz and $R[[T,\upsilon]]$ is $(S,\bar{\omega})$-Armendariz,*

(5) *R is $(S \times T, \omega \times \upsilon)$-Armendariz.*

Proof. (1) $\Leftrightarrow$ (2) Assume (1). Then by Theorem 8.3.12, for any $s \in S$ and $t \in T$ the endomorphisms ω_s and υ_t are rigid, and to get (2) it suffices to show that $\bar{\upsilon}_t$ is a rigid endomorphism of $R[[S,\omega]]$ for every $t \in T$. To prove that, consider any $f \in R[[S,\omega]]$ with $f\bar{\upsilon}_t(f) = 0$. Since R is (S,ω)-Armendariz, for any $s \in S$ we have $f(s) \cdot \omega_s(\bar{\upsilon}_t(f)(s)) = 0$, i.e. $f(s) \cdot \omega_s(\upsilon_t(f(s))) = 0$. Since ω_s and υ_t are rigid, it follows that $f(s)^2 = 0$, and since R is reduced, we obtain $f(s) = 0$. Thus $f = 0$, which completes the proof of (1) $\Rightarrow$ (2). The converse follows directly from Theorem 8.3.12 and Lemma 8.1.7.

(1) $\Leftrightarrow$ (3) This follows by an analogous argument.

(1) $\Leftrightarrow$ (4) As noted above, (1) implies (2) and (3), so it implies (4) as well. To prove the converse, assume (4). Since R is a subring of the $(S,\bar{\omega})$-Armendariz ring $R[[T,\upsilon]]$, Lemma 8.1.7 implies that R is (S,ω)-Armendariz. By a similar argument, R is (T,υ)-Armendariz.

(1) $\Leftrightarrow$ (5) If (1) holds, then by Theorem 8.3.12 the endomorphisms ω_s and υ_t are rigid for all $s \in S$ and $t \in T$. Hence for any $(s,t) \in S \times T$ the endomorphism

$(\omega \times \upsilon)_{(s,t)} \in \mathrm{End}(R)$ is rigid, and applying Theorem 8.3.12 we obtain (5). The implication (5) $\Rightarrow$ (1) is an immediate consequence of Lemma 8.1.7. $\square$

Corollary 8.4.8. *Let R be a reduced ring, $(S, \leqslant_S)$ and $(T, \leqslant_T)$ strictly ordered a.n.u.p. monoids, and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism such that ω_s is injective for every $s \in S$. Then R is (S,ω)-Armendariz if and only if $R[[T,1]]$ is $(S,\bar{\omega})$-Armendariz.*

Proof. Applying the implications (vi) $\Rightarrow$ (ii) and (vi) $\Rightarrow$ (iv) of Theorem 8.3.12, we deduce that R is $(T,1)$-Armendariz and $R[[T,1]]$ is reduced. Now the corollary follows from the equivalence (i) $\Leftrightarrow$ (iii) of Proposition 8.4.7. $\square$

Applying Corollary 8.4.8 when T is the additive monoid of nonnegative integers with the trivial order, the additive monoid of nonnegative integers with the usual order, the additive group of integers with the trivial order and the additive group of integers with the usual order, respectively, we obtain the following:

Corollary 8.4.9. *Let R be a reduced ring, $(S, \leqslant)$ a strictly ordered a.n.u.p. monoid, and $\omega\colon S \to \mathrm{End}(R)$ a monoid homomorphism such that ω_s is injective for every $s \in S$. Then the following are equivalent:*

(1) *The ring R is (S,ω)-Armendariz.*

(2) *The polynomial ring $R[x]$ is $(S,\bar{\omega})$-Armendariz.*

(3) *The power series ring $R[[x]]$ is $(S,\bar{\omega})$-Armendariz.*

(4) *The Laurent polynomial ring $R[x,x^{-1}]$ is $(S,\bar{\omega})$-Armendariz.*

(5) *The Laurent series ring $R[[x,x^{-1}]]$ is $(S,\bar{\omega})$-Armendariz.*

A special case of Corollary 8.4.9 is the following result of W. Chen and W. Tong (with a slight change in notation).

Corollary 8.4.10 ([12, Proposition 6]). *Let R be a reduced ring and σ a monomorphism of R. Then R is σ-skew Armendariz if and only if $R[x]$ is $\bar{\sigma}$-skew Armendariz.*

8.5 Chain rings are Armendariz

Proposition 8.5.1. *Let R be a right chain ring, $(S, \leqslant)$ a strictly ordered a.n.u.p. monoid and $\omega\colon S \to \operatorname{End}(R)$ a monoid homomorphism. Assume the Jacobson radical* $\operatorname{rad}(R)$ *is S-invariant. Suppose that $f, g \in R[[S, \omega]]$ satisfy $fg = 0$ and that there exist $s_0, t_0 \in S$ such that $f(s) \cdot \omega_s(g(t)) \in f(s_0) \cdot \omega_{s_0}(g(t_0))R$ for all $s, t \in S$. Then $f(s) \cdot \omega_s(g(t)) = 0$ for all $s, t \in S$.*

Proof. Let

$$\begin{aligned} X &= \{x \in S : (\exists u \in S)\,(\forall s, t \in S) \quad f(s) \cdot \omega_s(g(t)) \in f(x) \cdot \omega_x(g(u))R\} \\ Y &= \{y \in S : (\exists v \in S)\,(\forall s, t \in S) \quad f(s) \cdot \omega_s(g(t)) \in f(v) \cdot \omega_v(g(y))R\}. \end{aligned}$$

If $X \nsubseteq \operatorname{supp}(f)$ or $Y \nsubseteq \operatorname{supp}(g)$, then clearly $f(s) \cdot \omega_s(g(t)) = 0$ for all $s, t \in S$. Thus, we will assume that $X \subseteq \operatorname{supp}(f)$ and $Y \subseteq \operatorname{supp}(g)$. The sets X and Y are artinian, narrow, and nonempty (because $s_0 \in X$ and $t_0 \in Y$). Since $(S, \leqslant)$ is a.n.u.p., there exist $x_0 \in X$ and $y_0 \in Y$ such that $x_0 y_0$ is a u.p. element of XY. Since $x_0 \in X$ and $y_0 \in Y$, for some $u_0, v_0 \in S$ and all $s, t \in S$ we have

$$f(s) \cdot \omega_s(g(t)) \in f(x_0) \cdot \omega_{x_0}(g(u_0))R \tag{8.5.1}$$

and

$$f(s) \cdot \omega_s(g(t)) \in f(v_0) \cdot \omega_{v_0}(g(y_0))R. \tag{8.5.2}$$

Assume that $g(u_0)R \subseteq g(y_0)R$. Then $\omega_{x_0}(g(u_0))R \subseteq \omega_{x_0}(g(y_0))R$, and (8.5.1) implies that

$$f(s) \cdot \omega_s(g(t)) \in f(x_0) \cdot \omega_{x_0}(g(y_0))R \qquad \text{for all } s, t \in S. \tag{8.5.3}$$

Since $fg = 0$, we have

$$0 = (fg)(x_0 y_0) = f(x_0) \cdot \omega_{x_0}(g(y_0)) + \sum_{(p,q) \in X_{x_0 y_0}(f,g) \setminus \{(x_0, y_0)\}} f(p) \cdot \omega_p(g(q)). \tag{8.5.4}$$

Note that if $(p, q) \in S \times S \setminus \{(x_0, y_0)\}$ satisfies $pq = x_0y_0$, then since x_0y_0 is a u.p. element of XY, we have $p \notin X$ or $q \notin Y$. In either case, (8.5.3) implies that $f(x_0) \cdot \omega_{x_0}(g(y_0))R \nsubseteq f(p) \cdot \omega_p(g(q))R$. Therefore, since R is right chain,

$$f(p) \cdot \omega_p(g(q)) \in f(x_0) \cdot \omega_{x_0}(g(y_0)) \cdot \mathrm{rad}(R). \tag{8.5.5}$$

Thus, (8.5.4) and (8.5.5) imply that for some $r \in \mathrm{rad}(R)$ we have

$$0 = f(x_0) \cdot \omega_{x_0}(g(y_0)) + f(x_0) \cdot \omega_{x_0}(g(y_0))r = f(x_0) \cdot \omega_{x_0}(g(y_0))(1 + r),$$

which implies $f(x_0) \cdot \omega_{x_0}(g(y_0)) = 0$. By (8.5.3), $f(s) \cdot \omega_s(g(t)) = 0$ for all $s, t \in S$ in this case.

We are left with the case where $g(u_0)R \nsubseteq g(y_0)R$. Since R is right chain, $g(y_0)R \subseteq g(u_0) \cdot \mathrm{rad}(R)$. Choose $r \in \mathrm{rad}(R)$ such that $g(y_0) = g(u_0)r$. Then

$$\omega_{v_0}(g(y_0)) = \omega_{v_0}(g(u_0)) \cdot \omega_{v_0}(r) \in \omega_{v_0}(g(u_0)) \cdot \mathrm{rad}(R),$$

and from (8.5.2) we obtain

$$f(s) \cdot \omega_s(g(t)) \in f(v_0) \cdot \omega_{v_0}(g(u_0)) \cdot \mathrm{rad}(R) \qquad \text{for all } s, t \in S. \tag{8.5.6}$$

Applying (8.5.6) with $s = v_0$ and $t = u_0$, we obtain $f(v_0) \cdot \omega_{v_0}(g(u_0)) = 0$, and another application of (8.5.6) yields $f(s) \cdot \omega_s(g(t)) = 0$ for all $s, t \in S$, which completes the proof. $\square$

Corollary 8.5.2. *Let R be a right or left chain ring and S a u.p. monoid. Then R is Armendariz relative to S.*

Corollary 8.5.3. *Every right or left chain ring is Armendariz.*

Corollary 8.5.4. *Let R be a right noetherian, right chain ring, $(S, \leqslant)$ a strictly ordered a.n.u.p. monoid, and $\omega \colon S \to \mathrm{End}(R)$ a monoid homomorphism. Assume the Jacobson radical $\mathrm{rad}(R)$ is S-invariant. Then R is (S, ω)-Armendariz.*

Recall that for a commutative ring R an R-module M is called *divisible* if for every $r \in R$ which is not zero-divisor, and for every $m \in M$ there exists an element $n \in M$ such that $m = rn$. An R-module M is called *torsion-free* if for every $r \in R\backslash\{0\}$ and $m \in M\backslash\{0\}$, $rm \neq 0$. Recall that a right R-module M is said to be *chain* if the submodule lattice of M is totally ordered.

The following example shows that in Corollary 8.5.4 the noetherian hypothesis is essential.

Example 8.5.5. Let U be a commutative chain domain and M a divisible, chain U-module that is not torsion-free. Such a pair U and M exist under Zermelo–Fraenkel set theory with the axiom of choice by [19, Lemma 7]. Choose $u \in U\backslash\{0\}$ and $m \in M \setminus \{0\}$ such that $um = 0$. Put $m_0 = m$. Then by divisibility of M we can define a sequence $\{m_0, m_1, m_2, \ldots\}$ of elements of M such that $m_{n-1} = um_n$ for all $n \in \mathbb{N}$. Define $R = U \oplus M$ as an additive group, with multiplication given by

$$(v_1, n_1)(v_2, n_2) = (v_1v_2, v_1n_2 + v_2n_1).$$

Since M is divisible and U and M are chain, it follows that R is a commutative chain ring. Nevertheless, R is not $(S, 1)$-Armendariz for $S = \mathbb{N} \cup \{0\}$ with the standard order $\leqslant$. Indeed, this ring $R[[S, 1]]$ is isomorphic to the power series ring $R[[x]]$, and for the power series $f, g \in R[[x]]$ defined by

$$f = (u, 0) - (1, 0)x, \qquad g = (0, m_0) + (0, m_1)x + (0, m_2)x^2 + \cdots,$$

we have $fg = 0$, although $(1, 0)(0, m_0) \neq 0$ in R. $\square$

8.6 Triangular matrix rings

In [30], Hong, Kim, and Kwak obtained a wide range of detailed results on the skew Armendariz condition in triangular matrix rings. We will now prove a proposition that unifies two of the results in [30] within the context of skew generalized power series rings.

Let R be a ring, S a monoid, ω: $S \to \mathrm{End}(R)$ a monoid homomorphism, n a positive integer, and $\mathbb{M}_n(R)$ the ring of n by n matrices over R. For $s \in S$, let $\overline{\omega}_s$: $\mathbb{M}_n(R) \to \mathbb{M}_n(R)$ be the map obtained by applying ω_s to every entry of a given matrix in $\mathbb{M}_n(R)$. We thereby obtain a monoid homomorphism $\overline{\omega}$: $S \to \mathrm{End}(\mathbb{M}_n(R))$. Given any subring $T \subseteq \mathbb{M}_n(R)$ that is S-invariant, we have a monoid homomorphism, which (slightly abusing notation) we will also denote by $\overline{\omega}$: $S \to \mathrm{End}(T)$, obtained by restricting the homomorphisms $\overline{\omega}_s$ to T.

Proposition 8.6.1. *Let R be a ring, $(S, \leqslant)$ a strictly ordered monoid, ω: $S \to \mathrm{End}(R)$ a monoid homomorphism, and n any positive integer.*

Define a subring T of $\mathbb{M}_n(R)$ as follows:

$$T = \left\{ \begin{pmatrix} a & b_1 & b_2 & b_3 & \cdots & b_{n-1} \\ 0 & a & c_1 & c_2 & \cdots & c_{n-2} \\ 0 & 0 & a & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a & 0 \\ 0 & 0 & \cdots & 0 & 0 & a \end{pmatrix} : a, b_1, \ldots, b_{n-1}, c_1, \ldots, c_{n-2} \in R \right\}.$$

Suppose R is reduced, and ω_s is injective for every $s \in S$. Then R is (S, ω)-Armendariz if and only if T is $(S, \overline{\omega})$-Armendariz.

Proof. If T is $(S, \overline{\omega})$-Armendariz then R is (S, ω)-Armendariz by Lemma 8.1.7.

Conversely, assume R is (S, ω)-Armendariz. Suppose $f, g \in T[[S, \overline{\omega}]]$ satisfy $fg = 0$. Now, f and g are functions from S to T with artinian, narrow support. Given any $s \in S$, we have

$$f(s) = \begin{pmatrix} f_{1,1}(s) & f_{1,2}(s) & f_{1,3}(s) & f_{1,4}(s) & \cdots & f_{1,n}(s) \\ 0 & f_{2,2}(s) & f_{2,3}(s) & f_{2,4}(s) & \cdots & f_{2,n}(s) \\ 0 & 0 & f_{3,3}(s) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & f_{n-1,n-1}(s) & 0 \\ 0 & 0 & \cdots & 0 & 0 & f_{n,n}(s) \end{pmatrix}$$

and

$$g(s) = \begin{pmatrix} g_{1,1}(s) & g_{1,2}(s) & g_{1,3}(s) & g_{1,4}(s) & \cdots & g_{1,n}(s) \\ 0 & g_{2,2}(s) & g_{2,3}(s) & g_{2,4}(s) & \cdots & g_{2,n}(s) \\ 0 & 0 & g_{3,3}(s) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & g_{n-1,n-1}(s) & 0 \\ 0 & 0 & \cdots & 0 & 0 & g_{n,n}(s) \end{pmatrix}$$

where each $f_{i,j}$ and each $g_{i,j}$ is a function from S to R, and $f_{1,1} = f_{2,2} = \cdots = f_{n,n}$ and $g_{1,1} = g_{2,2} = \cdots = g_{n,n}$. Since $\mathrm{supp}(f_{i,j}) \subseteq \mathrm{supp}(f)$ and $\mathrm{supp}(g_{i,j}) \subseteq \mathrm{supp}(g)$, each $f_{i,j}$ and each $g_{i,j}$ has artinian, narrow support. Hence $f_{i,j},\ g_{i,j} \in R[[S,\omega]]$.

For every $s \in S$ we have

$$0 = (fg)(s) = \sum_{(x,y)\in \mathrm{X}_s(f,g)} f(x) \cdot \overline{\omega}_x(g(y)),$$

and therefore, for all i, j,

$$\begin{aligned} 0 &= \sum_k \sum_{(x,y)\in X_s(f,g)} f_{i,k}(x)\cdot\omega_x(g_{k,j}(y)) \\ &= \sum_k \sum_{(x,y)\in X_s(f_{i,k},g_{k,j})} f_{i,k}(x)\cdot\omega_x(g_{k,j}(y)) = \sum_k (f_{i,k}\, g_{k,j})(s). \end{aligned}$$

In the ring $R[[S,\omega]]$, which by Theorem 8.3.12 is reduced, the following equations hold:

$$f_{1,1}\, g_{1,1} = 0, \tag{8.6.1}$$

$$f_{1,1}\, g_{1,2} + f_{1,2}\, g_{1,1} = 0, \tag{8.6.2}$$

$$f_{1,1}\, g_{2,i} + f_{2,i}\, g_{1,1} = 0 \quad \text{for } i = 3, 4, \ldots, n, \tag{8.6.3}$$

$$f_{1,1}\, g_{1,i} + f_{1,2}\, g_{2,i} + f_{1,i}\, g_{1,1} = 0 \quad \text{for } i = 3, 4, \ldots, n. \tag{8.6.4}$$

Reduced rings are *symmetric* in the terminology of [50]: whenever a product of elements equals 0, any permutation of the factors also has product 0. Equations (8.6.1) and (8.6.2) therefore yield $0 = f_{1,1}\, g_{1,2}\, f_{1,1} + f_{1,2}\, g_{1,1}\, f_{1,1} = f_{1,1}\, g_{1,2}\, f_{1,1}$, and $(f_{1,1}\, g_{1,2})^2 = 0$ implies $f_{1,1}\, g_{1,2} = f_{1,2}\, g_{1,1} = 0$. Applying the same argument to equation (8.6.3) yields $f_{1,1}\, g_{2,i} = f_{2,i}\, g_{1,1} = 0$, and then applying it to equation (8.6.4) yields $f_{1,1}\, g_{1,i} = f_{1,2}\, g_{2,i} + f_{1,i}\, g_{1,1} = 0$. Since $f_{1,2}\, g_{1,1} = 0$, from $g_{1,1}\, f_{1,2}\, g_{2,i} + g_{1,1}\, f_{1,i}\, g_{1,1} = 0$ we likewise obtain $f_{1,i}\, g_{1,1} = f_{1,2}\, g_{2,i} = 0$. Thus, every summand in equations (8.6.1), (8.6.2), (8.6.3), and (8.6.4) equals 0 in $R[[S,\omega]]$.

By hypothesis, R is (S,ω)-Armendariz. Therefore, for all $s, t \in S$, we have

$$\begin{aligned} f_{1,1}(s)\cdot\omega_s(g_{1,1}(t)) &= 0, \\ f_{1,1}(s)\cdot\omega_s(g_{1,2}(t)) &= f_{1,2}(s)\cdot\omega_s(g_{1,1}(t)) = 0, \\ f_{1,1}(s)\cdot\omega_s(g_{2,i}(t)) &= f_{2,i}(s)\cdot\omega_s(g_{1,1}(t)) = 0 \quad \text{for } i = 3, 4, \ldots, n, \\ f_{1,1}(s)\cdot\omega_s(g_{1,i}(t)) &= f_{1,2}(s)\cdot\omega_s(g_{2,i}(t)) = f_{1,i}(s)\cdot\omega_s(g_{1,1}(t)) = 0 \\ &\qquad \text{for } i = 3, 4, \ldots, n. \end{aligned}$$

In particular, for all i, j,

$$\sum_k f_{i,k}(s) \cdot \omega_s(g_{k,j}(t)) = 0$$

and therefore $f(s) \cdot \overline{\omega}_s(g(t)) = 0$ for all $s, t \in S$. This proves that T is $(S, \overline{\omega})$-Armendariz. $\square$

The $n = 2$ case of the "only if" part of Proposition 8.6.1, in conjunction with Example 8.1.2(b), recovers [30, Proposition 15]. Analogously, the $n = 3$ case recovers [30, Proposition 17]. The fact that when $n \leqslant 3$ the ring of upper triangular n by n matrices with constant diagonal over a σ-rigid ring is $\overline{\sigma}$-skew Armendariz, as pointed out by Hong, Kim, and Kwak in [30, Example 18], does not generalize to $n = 4$. The fatal flaw can be traced to the nonzero $(3, 4)$-entry of the matrix! Proposition 8.6.1 demonstrates a different direction in which a viable generalization is possible.

In the proof of the next result we will need the following criterion for S-rigidity of subrings of an (S, ω)-Armendariz ring.

Lemma 8.6.2. *Let T be a ring, $(S, \leqslant)$ an ordered monoid, and $\omega\colon S \to \mathrm{End}(T)$ a monoid homomorphism. Suppose T is linearly (S, ω)-Armendariz, suppose R is an S-invariant, S-compatible subring of T, and suppose there exists $b \in T$ with the property that $b \neq b^2 = 0$, $R \cap r_T(b) = \{0\}$, and there exists $s \in S \setminus \{1\}$ such that for every $r \in R$ we have $br = r\omega_s(b)$. Then R is S-rigid.*

Proof. By Lemma 8.1.5(iii), it suffices to show R is reduced. Suppose $a \in R$ satisfies $a^2 = 0$. Put

$$f = \mathsf{c}_b + \mathsf{c}_a \mathsf{e}_s, \quad g = \mathsf{c}_b + \mathsf{c}_{-a} \mathsf{e}_s \quad \text{in} \quad T[[S, \omega]].$$

Using the S-compatibility of R and the hypotheses on b, we find that $fg = 0$. Since T is linearly (S, ω)-Armendariz, $-ba = 0$, hence $a \in R \cap r_T(b) = \{0\}$. $\square$

Corollary 8.6.3. *Let R be a ring, $(S, \leqslant)$ a nontrivial strictly ordered monoid, $\omega \colon S \to \mathrm{End}(R)$ a monoid homomorphism, and $n \geqslant 2$ an integer. Let $T \subset \mathbb{M}_n(R)$ be the subring defined in Proposition 8.6.1. Suppose R is S-compatible. Then R is reduced and (S, ω)-Armendariz if and only if T is $(S, \overline{\omega})$-Armendariz.*

Proof. To prove "only if" part we can apply Lemma 8.1.5(i) and Proposition 8.6.1. The "if" part follows from Lemma 8.6.2 (b can be taken to be the matrix with a 1 in the $(1,2)$-position and 0's elsewhere) and Lemma 8.1.7. □

Corollary 8.6.3 shows that the hypothesis in Proposition 8.6.1 that R be reduced is indispensable. Clearly, the conclusion of Proposition 8.6.1 fails without the hypothesis that every ω_s be injective. So Proposition 8.6.1 is "sharp" in some sense.

Chapter 9

A new class of unique product monoids

This Chapter is based on:

• G. Marks, R. Mazurek, M. Ziembowski, *A new class of unique product monoids with applications to ring theory*, Semigroup Forum 78 (2009), 210–225.

In this chapter we will give details, examples, and interrelationships between some subclasses of unique product monoids.

9.1 Artinian narrow unique product monoids

At the beginning of the present section we would like to recall our central definition of the class of unique product monoids.

Definition 9.1.1. Let $(S, \leqslant)$ be an ordered monoid. We say that $(S, \leqslant)$ is an *artinian narrow unique product monoid* (or an *a.n.u.p. monoid*, or simply *a.n.u.p.*) if for every two artinian and narrow subsets A and B of S there exists a u.p. element in the product AB.

Clearly, every finite subset of an ordered set is artinian and narrow, and thus all a.n.u.p. monoids are indeed u.p. monoids. It is also clear that u.p. monoids are exactly a.n.u.p. monoids with respect to the *trivial order*, i.e. the order with respect to which any two distinct elements are incomparable.

In this section we relate the new class of a.n.u.p. monoids to the well established classes of totally ordered monoids and u.p. monoids, as well as to the natural extensions of totally ordered monoids defined below. For a partially ordered set X we write $\min X$ to denote the set of minimal elements of X.

Definition 9.1.2. A monoid S is said to be *totally orderable* if $(S, \leqslant)$ is an ordered monoid for some total order $\leqslant$. We say that an ordered monoid $(S, \leqslant)$ is a *minimal artinian narrow unique product monoid* (or a *m.a.n.u.p. monoid*, or simply *m.a.n.u.p.*) if for every two artinian and narrow subsets A and B of S there exist $a \in \min A$ and $b \in \min B$ such that ab is a u.p. element of AB.

Remark 9.1.3. (Cf. [63, Remark 7 in Chapter 10].) For cancellative monoids (though not for monoids in general!), one could replace products of artinian and narrow subsets by "squares" of artinian and narrow subsets in the definitions of *a.n.u.p.* and *m.a.n.u.p.* More precisely, we have the following observation:

Lemma 9.1.4. *Let $(S, \leqslant)$ be an ordered monoid.*

(i) *$(S, \leqslant)$ is an a.n.u.p. monoid if and only if S is cancellative and for every nonempty artinian and narrow subset $C \subseteq S$ there exists a u.p. element in the product CC.*

(ii) *$(S, \leqslant)$ is a m.a.n.u.p. monoid if and only if S is cancellative and for every nonempty artinian and narrow subset $C \subseteq S$ there exist $c_1, c_2 \in \min C$ such that c_1c_2 is a u.p. element of CC.*

Proof. The proof of (i) is subsumed by that of (ii), so we will prove the latter. The "only if" implication is obvious. To prove the converse, suppose A and B are artinian and narrow subsets of S. It follows from [27, Theorem 2.1] that $C = BA$ is artinian and narrow. By hypothesis, there exist $c_1, c_2 \in \min C$ such that c_1c_2 is a u.p. element of CC. Write $c_1 = b_1a_1$ and $c_2 = b_2a_2$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$. If $a_1 \notin \min A$, then $a_0 < a_1$ for some $a_0 \in A$, and since S

is a cancellative ordered monoid we obtain $b_1a_0 < c_1$, a contradiction. Hence $a_1 \in \min A$. Similarly, $b_2 \in \min B$. To complete the proof, we will show that a_1b_2 is a u.p. element of AB. If for some $a \in A$ and $b \in B$ we have $a_1b_2 = ab$, then from $c_1c_2 = (b_1a_1)(b_2a_2) = (b_1a)(ba_2)$ we obtain $b_1a_1 = b_1a$ and $b_2a_2 = ba_2$. Since S is cancellative, $a_1 = a$ and $b_2 = b$. □

If $(S, \leqslant)$ is an ordered monoid, then the following implications hold:

$$
\begin{array}{c}
S \text{ is a commutative, torsion-free, cancellative monoid} \\
\Downarrow \\
(S, \leqslant) \text{ is quasitotally ordered} \\
\Downarrow \\
(S, \leqslant) \text{ is a m.a.n.u.p. monoid} \\
\Downarrow \\
(S, \leqslant) \text{ is an a.n.u.p. monoid} \\
\Downarrow \\
S \text{ is a u.p. monoid}
\end{array}
$$

The top implication is a well-known result (e.g. see [68, 3.3]; note that the torsion-free assumption is quite natural—indeed, a sine qua non—in the motivating context of the Zero-Divisor Conjecture for group rings). To prove the second implication note that by assumption $\leqslant$ can be refined to a total order $\preccurlyeq$ such that $(S, \preccurlyeq)$ is a strictly ordered monoid. If A and B are artinian and narrow subsets of $(S, \leqslant)$, then the sets $\min A$ and $\min B$ are finite, and thus we can choose elements $a \in \min A$ and $b \in \min B$ which are smallest under the total order $\preccurlyeq$. Now it is easy to see that ab is a u.p.-element of AB. The remaining implications are obvious.

If the order $\leqslant$ on S is trivial, the bottom two implications in the above diagram become equivalences. This diagram of implications might be contrasted with that on p. 591 of D. S. Passman's classic monograph [64]. It is worth noting that one of the arrows in [64, p. 591] has a "surprise converse" (which, in turn, enables us to remove a hypothesis on the characteristic of a ground field from another one of the arrows in [64, p. 591]). Namely, A. Strojnowski proved in [73] that the so-called *t.u.p. groups* are precisely the u.p. groups (see Remark 9.1.13). This revelation suggests that it will be worthwhile to prove that all of the implications in our diagram above are irreversible, at least in the cases where this is less than obvious. This we will do in Examples 9.1.7, 9.1.8, and 9.1.11, to follow.

To construct our examples, we introduce the following method of defining strict orders on monoids.

Construction 9.1.5. *Let S be a monoid and $\mathcal{X}$ a subset of $S \times S$. For $u_1, u_2 \in S$, we write $u_1 \curvearrowright u_2$ if there exist $(x, y) \in \mathcal{X}$ and $v, w \in S$ such that*

$$u_1 = vxw \quad and \quad u_2 = vyw.$$

For $s, t \in S$, we write $s \leqslant t$ if $s = t$ or there exists a finite set of elements $u_0, u_1, \ldots, u_n \in S$ such that:

(i) $u_0 = s$ and $u_n = t$, and

(ii) $u_i \curvearrowright u_{i+1}$ for every $i \in \{0, 1, \ldots, n-1\}$.

Note that any order on S can be obtained by this method. Namely, if $(S, \leqslant)$ is an ordered monoid, then starting with the set $\mathcal{X} = \{(a, b) \in S \times S : a \leqslant b\}$, Construction 9.1.5 recovers the order $\leqslant$.

Lemma 9.1.6. *In Construction 9.1.5, suppose that for any finite set of elements $u_0, u_1, \ldots, u_n \in S$ such that $u_i \curvearrowright u_{i+1}$ for each $i \in \{0, 1, \ldots, n-1\}$ we have $u_0 \neq u_n$. Then $(S, \leqslant)$ is a strictly ordered monoid.*

Proof. Easy to verify. The hypotheses ensure that the relation $\leqslant$ is antisymmetric and the order $\leqslant$ is strict. □

Example 9.1.7. *A m.a.n.u.p. monoid that is not quasitotally ordered.* Let S be the free monoid on $\{s, t\}$, and let $\mathcal{X} = \{(st, t^2), (ts, s^2)\}$. We will show that the condition in Lemma 9.1.6 holds. If not, then there exists a finite sequence of elements $u_0, u_1, \dots, u_n \in S$ such that $u_i \curvearrowright u_{i+1}$ for each $i \in \{0, 1, \dots, n-1\}$, and $u_n = u_0$. Choose such a sequence for which the length m of u_0, considered as a word in the letters s and t, is minimal.

For every $i \in \{0, 1, \dots, n-1\}$ there exist $v_i, w_i \in S$ and $(x_i, y_i) \in \mathcal{X}$ such that $u_i = v_i x_i w_i$ and $u_{i+1} = v_i y_i w_i$. We consider the following two cases.

Case 1. *$w_i \neq 1$ for every $i \in \{0, 1, \dots, n-1\}$. Then for every i there exists $w_i' \in S$ such that $w_i = w_i's$ or $w_i = w_i't$. Since $v_{i-1}y_{i-1}w_{i-1} = u_i = v_i x_i w_i$ for any $i \geqslant 1$, it follows that if $w_0 = w_0's$ (resp. $w_0 = w_0't$), then $w_i = w_i's$ (resp. $w_i = w_i't$) for every i. Putting $u_i' = v_i x_i w_i'$ gives us a new finite sequence $u_0', u_1', u_2', \dots, u_n'$ with $u_i' \curvearrowright u_{i+1}'$ for every $i \in \{0, 1, \dots, n-1\}$, and $u_n' = u_0'$. But the length of u_0' is $m - 1 < m$, a contradiction.*

Case 2. *$w_i = 1$ for some $i \in \{0, 1, \dots, n-1\}$. Applying the automorphism of S that interchanges s and t if necessary, we can assume without loss of generality that $x_i = st$ and $y_i = t^2$. Thus $u_{i+1} = v_i t^2$. It is easy to see that for any $a, b \in S$, if $a \curvearrowright b$ and $a = a_1 t^2$ for some $a_1 \in S$, then $b = b_1 t^2$ for some $b_1 \in S$. This and the equality $u_n = u_0$ imply that for any $j \in \{0, 1, \dots, n\}$ there exists $z_j \in S$ such that $u_j = z_j t^2$, which leads to $z_i t^2 = u_i = v_i st$, a contradiction.*

By the above, $(S, \leqslant)$ is strictly ordered. Suppose $\leqslant$ is a quasitotal order, i.e. $\leqslant$ can be refined to a total order $\preccurlyeq$ with respect to which S is an ordered monoid. Since $st < t^2$, also $st \prec t^2$, and thus $t \preccurlyeq s$ is impossible. Similarly $ts < s^2$ eliminates $s \preccurlyeq t$. Thus neither $t \preccurlyeq s$ nor $s \preccurlyeq t$, a contradiction.

To make the example complete, we need to show that $(S, \leqslant)$ is a m.a.n.u.p. monoid. Suppose A and B are any two nonempty subsets of S (it does not matter if they are artinian and narrow). Let a be a word in $\min A$ of minimal length (considered as a word in the letters s and t), and b a word in $\min B$ of minimal length. Clearly, for any $u_1, u_2 \in S$, $u_1 \curvearrowright u_2$ implies that u_1 and u_2 are of the same length, and thus no member of A is shorter than a, and no member of B is shorter than b. Therefore, if $ab = a_1 b_1$ for some $a_1 \in A$ and $b_1 \in B$, then

a and a_1 are of the same length, and b and b_1 are of the same length. Since S is a free monoid, it follows that $a = a_1$ and $b = b_1$. $\square$

Our next example will show that a monoid can be totally ordered with respect to one order, but not a.n.u.p. with respect to another. In particular, *u.p.* does not imply *a.n.u.p.*

Example 9.1.8. *A totally orderable monoid that is not a.n.u.p.* Let S be the monoid generated by $\{x_i : i \in \mathbb{N}\} \cup \{X_j : j \in \mathbb{N}\}$ with the following relations:

$$x_i X_j = \begin{cases} x_{i-2} X_{i-2} & \text{if } i \geqslant 3 \text{ and } j = i + (-1)^{i+1} \\ x_j X_i & \text{otherwise.} \end{cases}$$

Hence $x_i X_j = x_j X_i$ for any $i \neq j$ except for the following products:

$$x_3 X_4 = x_1 X_1, \ x_4 X_3 = x_2 X_2, \ x_5 X_6 = x_3 X_3, \ x_6 X_5 = x_4 X_4, \ \text{and so on.}$$

The *length* of an element of S will mean its length as a word in $\{x_i : i \in \mathbb{N}\} \cup \{X_j : j \in \mathbb{N}\}$ (note that this length is well-defined).

First we show that, with respect to an appropriate order $\preccurlyeq$, S is a totally ordered monoid. Let U be the set of all elements of S of the form $x_i X_j$. Observe that by the defining relations every element $u \in U$ can be written in the form $x_i X_j$ in exactly two ways; the form that minimizes the value of j will be called *normal* and denoted by $[u]$. Thus,

$$[x_i X_j] = \begin{cases} x_{i-2} X_{i-2} & \text{if } i \geqslant 3 \text{ and } j = i + (-1)^{i+1} \\ x_{\max\{i,j\}} X_{\min\{i,j\}} & \text{otherwise.} \end{cases}$$

We extend the notion of the normal form to all elements of S in the obvious way. Namely, let T be the submonoid of S generated by $\{x_i : i \in \mathbb{N}\}$, let Z be the submonoid of S generated by $\{X_i : i \in \mathbb{N}\}$, and put $V = T \cup Z \cup ZT$ (i.e. V consists of the empty word, the words in x_i's, the words in X_i's, and the words which are products of X_i's followed by a product of x_i's). Then every element $s \in S$ can be written in the form

$$s = v_1 \qquad \text{or} \qquad s = v_1 u_1 v_2 u_2 \cdots v_{n-1} u_{n-1} v_n,$$

where each v_k belongs to V, and each u_k belongs to U. If s contains internal factors u_k, then the unique expression for s in which every internal factor u_k has been written as $[u_k]$ will be called the *normal form* of s. If s does not contain any internal factors u_k, then the unique expression for s as a (possibly empty) word in $\{x_i : i \in \mathbb{N}\} \cup \{X_j : j \in \mathbb{N}\}$ will be called the *normal form* of s.

Now we are ready to define a total order on S. We start by ordering the generators of S as follows:

$$\cdots \prec x_6 \prec x_4 \prec x_2 \prec x_1 \prec x_3 \prec x_5 \prec \cdots \prec X_6 \prec X_4 \prec X_2 \prec X_1 \prec X_3 \prec X_5 \prec \cdots.$$

We extend this order to all elements of S by declaring for $s, t \in S$ that $s \preccurlyeq t$ if and only if either $s = t$, or the length of s is less than the length of t, or s and t are of the same length and the normal form of s precedes the normal form of t in the lexicographical ordering.

To illustrate this definition, we compare the following three elements of S:

$$s_1 = X_5 x_6 x_4 X_3 X_5 x_5 X_2 x_5 X_6 X_8 X_3 x_1,$$

$$s_2 = x_4 X_1 x_5 X_8 x_9 x_2 X_1 x_3 x_3,$$

$$s_3 = X_5 x_6 x_2 X_2 X_5 x_2 X_5 x_3 X_3 X_3 x_4 X_7.$$

Since s_2 is shorter than s_1 and s_3, we have $s_2 \prec s_1$ and $s_2 \prec s_3$. Since s_1 and s_3 are of the same length, to compare these elements we write them in the normal form:

$$s_1 = X_5 x_6 x_2 X_2 X_5 x_5 X_2 x_3 X_3 X_8 X_3 x_1,$$

$$s_3 = X_5 x_6 x_2 X_2 X_5 x_5 X_2 x_3 X_3 X_3 x_7 X_4.$$

Now we look at the first place from the left where the normal forms of s_1 and s_3

differ. Since $X_8 \prec X_3$, it follows that $s_1 \prec s_3$. Thus $s_2 \prec s_1 \prec s_3$.

It is clear that the relation $\preccurlyeq$ is a total order on the set S. To prove that the order $\preccurlyeq$ is strict, it is enough to show that for all i, j, k,

$$x_i \prec x_j \implies x_iX_k \prec x_jX_k \qquad \text{and} \qquad X_i \prec X_j \implies x_kX_i \prec x_kX_j.$$

We will only prove the first implication; the proof of the second implication is analogous and thus left to the reader.

Assume that $x_i \prec x_j$. It follows from the definition of the order $\preccurlyeq$ that we cannot have i odd and j even, and furthermore, the following implications hold:

$$i \text{ and } j \text{ even} \Rightarrow i > j; \qquad i \text{ and } j \text{ odd} \Rightarrow i < j. \tag{$*$}$$

To prove that $x_iX_k \prec x_jX_k$, we consider four cases, depending upon the normal forms of x_iX_k and x_jX_k.

Case 1. *$[x_iX_k] = x_{i-2}X_{i-2}$ and $[x_jX_k] = x_{j-2}X_{j-2}$. Then $k = i + (-1)^{i+1}$ and $k = j + (-1)^{j+1}$. Hence, if i and j have opposite parity, then k is simultaneously odd and even, a contradiction. Otherwise it follows that $i = j$, and this contradiction shows that case 1 is impossible.*

Case 2. *$[x_iX_k] = x_{i-2}X_{i-2}$ and $[x_jX_k] = x_{\max\{j,k\}}X_{\min\{j,k\}}$. Then to prove that $x_iX_k \prec x_jX_k$, it suffices to show that*

$$x_{i-2} \prec x_{\max\{j,k\}}. \tag{1}$$

If i and j are odd, then $k = i+1$, and from ($*$) we deduce $i-2 < j = \max\{j,k\}$, which implies (1), since $i-2$ and j are odd. If i and j are even, then $k = i-1$; thus ($*$) implies that $\max\{j,k\} = k$ is odd, and since $i-2$ is even, (1) follows. Finally, if i is even and j is odd, then $k = i-1$ and thus $\max\{j,k\}$ is odd, whereas $i-2$ is even, proving (1).

Case 3. *$[x_iX_k] = x_{\max\{i,k\}}X_{\min\{i,k\}}$ and $[x_jX_k] = x_{j-2}X_{j-2}$. Then it can be shown* *analogously as in case 2 that if i and j are even, then* $\max\{i,k\} = i > j-2$, *and if j is odd, then* $\max\{i,k\}$ *is even. Hence* $x_{\max\{i,k\}} \prec x_{j-2}$, *which obviously implies that* $x_iX_k \prec x_jX_k$.

Case 4. $[x_iX_k] = x_{\max\{i,k\}}X_{\min\{i,k\}}$ *and* $[x_jX_k] = x_{\max\{j,k\}}X_{\min\{j,k\}}$. *If* $i \geqslant k$ *and* $j \geqslant k$, *then* $[x_iX_k] = x_iX_k$ *and* $[x_jX_k] = x_jX_k$, *and since* $x_i \prec x_j$, *we obtain* $x_iX_k \prec x_jX_k$. *If* $k \geqslant i$ *and* $k \geqslant j$, *then* $[x_iX_k] = x_kX_i$ *and* $[x_jX_k] = x_kX_j$, *and since* $x_i \prec x_j \Leftrightarrow X_i \prec X_j$, *it follows that* $x_iX_k \prec x_jX_k$. *If* $i > k > j$, *then* i *cannot be odd, and* i *even implies* $x_i \prec x_k$, *giving* $x_iX_k \prec x_jX_k$. *If* $j > k > i$, *then* j *cannot be even, and* $x_k \prec x_j$ *follows, proving* $x_iX_k \prec x_jX_k$.

Thus, $(S, \preccurlyeq)$ is a strictly totally ordered monoid (in particular, S is a u.p. monoid).

We will now define a strict order $\leqslant$ on S with respect to which S is not a.n.u.p. Put

$$\mathcal{X} = \{(x_i, x_{i+2}) : i \in \mathbb{N}\} \cup \{(X_i, X_{i+2}) : i \in \mathbb{N}\},$$

and define $\leqslant$ according to Construction 9.1.5.

To prove that $(S, \leqslant)$ is a strictly ordered monoid, we apply Lemma 9.1.6. Suppose $u_0, u_1, u_2, \ldots, u_n \in S$ satisfy $u_i \curvearrowright u_{i+1}$ for each $i \in \{0, 1, 2, \ldots, n-1\}$, where $n \in \mathbb{N}$. To prove $u_0 \neq u_n$, it is enough to consider the case where each u_k belongs to the set U of all elements of S of the form x_iX_j. As already observed, every element $u \in U$ can be uniquely written in the normal form x_iX_j. We can define the *height* of u as $h(u) = i + j$ where $[u] = x_iX_j$ is the normal form of u. Note that if we would have $u_0 = u_n$, then we would have an infinite periodic sequence

$$u_0 \curvearrowright u_1 \curvearrowright \ldots \curvearrowright u_{n-1} \curvearrowright u_n = u_0 \curvearrowright u_1 \curvearrowright \ldots \curvearrowright u_{n-1} \curvearrowright u_n = u_0 \curvearrowright \ldots.$$

Hence, using the periodicity of the sequence and removing some its beginning terms if necessary, we could assume that $n \geqslant 3$ and

$$h(u_0) = \max\{h(u_0), h(u_1), \ldots, h(u_n)\}.$$

But this would contradict the following claim.

Claim. *For any $u_0, u_1, u_2, u_3 \in U$, if $u_0 \curvearrowright u_1$, $u_1 \curvearrowright u_2$ and $u_2 \curvearrowright u_3$, then $h(u_0) < h(u_1)$ or $h(u_0) < h(u_2)$ or $h(u_0) < h(u_3)$.*

To prove the claim we first analyze relations between the heights $h(u)$ and $h(v)$ for $u, v \in U$ with $u \curvearrowright v$. There are four cases:

Case 1. *$[u] = x_i X_j$ with $i > j + 3$ or $i = j + 2$. Then either $u = x_i X_j$ or $u = x_j X_i$, and thus the only possibilities for the normal form of v are $x_{i+2} X_j$ and $x_i X_{j+2}$. Hence $h(v) = h(u) + 2$.*

Case 2. *$[u] = x_j X_j$. The only possibilities for the normal form of v are as follows:*

(a) *$x_{j+2} X_j$; then $h(v) = h(u) + 2$, and v falls under case 1;*

(b) *either $x_{j+3} X_{j+2}$ or $x_{j+4} X_{j+1}$ (both with even j); then $h(v) = h(u) + 5$;*

(c) *either $x_{j+4} X_{j+3}$ or $x_{j+5} X_{j+2}$ (both with odd j); then $h(v) = h(u) + 7$.*

Case 3. *$[u] = x_{j+3} X_j$. The only possibilities for the normal form of v are as follows:*

(a) *$x_{j+5} X_j$; then $h(v) = h(u) + 2$;*

(b) *$x_{j+3} X_{j+2}$ (with even j); then $h(v) = h(u) + 2$;*

(c) *either $x_j X_j$ or $x_{j+1} X_{j+1}$ (both with odd j); then either $h(v) = h(u) - 3$ or $h(v) = h(u) - 1$, and v falls under case 2.*

Case 4. *$[u] = x_{j+1} X_j$. The only possibilities for the normal form of v are as follows:*

(a) *$x_{j+3} X_j$; then $h(v) = h(u) + 2$;*

(b) *$x_{j+2} X_{j+1}$ (with odd j); then $h(v) = h(u) + 2$;*

(c) *either $x_{j-1} X_{j-1}$ or $x_j X_j$ (both with even j); then either $h(v) = h(u) - 3$ or $h(v) = h(u) - 1$, and v falls under case 2.*

Now we are ready to prove the claim. If $h(u_0) \geqslant h(u_1)$, then either $h(u_1) = h(u_0) - 3$ or $h(u_1) = h(u_0) - 1$ (case 3(c) or 4(c)), and in either case u_1 falls under case 2. Hence either $h(u_2) \geqslant h(u_1) + 5$ (if u_1 falls under case 2(b) or 2(c)), and then $h(u_2) > h(u_0)$, or else $h(u_2) = h(u_1) + 2$ and u_2 falls under case 1 (if u_1 falls under case 2(a)). But if u_2 falls under case 1, then $h(u_3) = h(u_2) + 2$, and thus $h(u_3) > h(u_0)$. The claim is proved.

Finally, we show that $(S, \leqslant)$ is not an a.n.u.p. monoid. Let $A = \{x_n : n \in \mathbb{N}\}$ and $B = \{X_n : n \in \mathbb{N}\}$. Since each element of A occurs in one of the following increasing sequences:

$$x_1 < x_3 < x_5 < \dots \quad \text{or} \quad x_2 < x_4 < x_6 < \dots,$$

it follows that A is artinian and narrow, and by the same argument so is the set B. From the defining relations it follows that there is no u.p. element in AB. □

We have seen in Example 9.1.7 that the implication "quasitotally ordered ⇒ m.a.n.u.p." is irreversible. In that example, the monoid was not quasitotally ordered with respect to the order under which it was m.a.n.u.p.; however, being a free monoid, it was totally ordered with respect to a different order. Our next example is of a m.a.n.u.p. monoid that is not totally ordered under any order.

Example 9.1.9. *A m.a.n.u.p. monoid that is not totally orderable.* We use [63, Example 13 of Chapter 10], due to Krempa. Let S be the monoid generated by $x_1, x_2, x_3, X_1, X_2, X_3$ subject to the following relations:

$$x_1X_1 = x_2X_3, \qquad x_1X_2 = x_3X_1, \qquad x_1X_3 = x_2X_2, \qquad x_3X_2 = x_2X_1.$$

As shown in [63], S is a u.p. monoid; thus, if $\leqslant$ is the trivial order on S, then $(S, \leqslant)$ is m.a.n.u.p. Suppose that $(S, \preccurlyeq)$ is a totally ordered monoid for some order $\preccurlyeq$. (Since S is a u.p. monoid, it is cancellative; therefore, $(S, \preccurlyeq)$ is necessarily strictly ordered.) Let $A = \{x_1, x_2, x_3\}$ and $B = \{X_1, X_2, X_3\}$. Let a (resp. b) be the minimal element of A (resp. B) under $\preccurlyeq$, and let c (resp. d) be the maximal element of A (resp. B) under $\preccurlyeq$. Then ab and cd are distinct u.p. elements of AB. But of the five elements of AB only one, x_3X_3, is a u.p. element. Hence S is not a totally ordered monoid, under any order. $\square$

We now turn to perhaps the most interesting of our non-implications: *a.n.u.p.* does not imply *m.a.n.u.p.* We first prove the following proposition, showing that in certain cases *a.n.u.p.* is equivalent to *u.p.*

Proposition 9.1.10. *Let $(M, \leqslant)$ and $(N, \preccurlyeq)$ be strictly ordered monoids. Let $\varphi\colon M \to N$ be a monoid homomorphism with the property that $|\varphi^{-1}(n)| < \infty$ for every $n \in N$, and assume, in addition, that for every artinian and narrow subset A of M, $\varphi(A)$ contains a unique minimal element. Then M is a u.p. monoid if and only if $(M, \leqslant)$ is an a.n.u.p. monoid.*

Proof. The "if" part is clear. For the "only if" part, let A and B be artinian, narrow subsets of M. Let n_a, $n_b \in N$ be the minimal elements of $\varphi(A)$, $\varphi(B)$ respectively, and put $A' = A \cap \varphi^{-1}(n_a)$ and $B' = B \cap \varphi^{-1}(n_b)$. Since M is a u.p. monoid, there exist $a \in A'$ and $b \in B'$ such that ab is a u.p. element of $A'B'$. We show that ab is also a u.p. element of AB. For let $c \in A$ and $d \in B$ be such that $ab = cd$. Then $n_an_b = \varphi(a)\varphi(b) = \varphi(c)\varphi(d)$, and since $(N, \preccurlyeq)$ is strictly ordered, it follows that $n_a = \varphi(c)$ and $n_b = \varphi(d)$. Hence $c \in A'$ and $d \in B'$, and thus the equality $ab = cd$ implies that $a = c$ and $b = d$. $\square$

Example 9.1.11. *An a.n.u.p. monoid need not be m.a.n.u.p.* Let $(S, \leqslant)$ be the ordered monoid constructed in Example 9.1.8, which we proved to be u.p. but not a.n.u.p. Let M be the submonoid of S generated by $\{x_1, x_2, x_3, x_4, X_1, X_2, X_3, X_4\}$. The order $\leqslant$ on S induces an order on M, which we will also denote by $\leqslant$. Then $(M, \leqslant)$ is a strictly ordered u.p. monoid.

Let N be the additive monoid of nonnegative integers, with $\preccurlyeq$ the standard order on N. Let $\varphi\colon M \to N$ be the monoid homomorphism determined by $\varphi(x_i) = \varphi(X_i) = 1$ for every i. Proposition 9.1.10 implies that $(M, \leqslant)$ is an a.n.u.p. monoid.

Put $A = \{x_1, x_2, x_3, x_4\}$ and $B = \{X_1, X_2, X_3, X_4\}$. Then $A_0 = \min A = \{x_1, x_2\}$ and $B_0 = \min B = \{X_1, X_2\}$. Since $x_1X_1 = x_3X_4$, $x_2X_2 = x_4X_3$, and $x_1X_2 = x_2X_1$, it follows that no element of A_0B_0 is a u.p. element of AB. Hence $(M, \leqslant)$ is not m.a.n.u.p. $\square$

Remark 9.1.12. In Example 9.1.11, $(M, \leqslant)$ satisfies an even stronger condition than *a.n.u.p.*; namely, for any two nonempty subsets A and B of M there exists a u.p. element of AB.

Remark 9.1.13. Recall that a monoid S is said to be a *two unique product monoid* (a *t.u.p. monoid* for short) if for any nonempty finite subsets A and B of S with $|A| + |B| > 2$ there exist at least two u.p. elements in AB. By a result of Strojnowski [73, Theorem 1], a group is t.u.p. if and only if it is u.p.

One could define an a.n.t.u.p. group in the obvious way: an ordered group $(S, \leqslant)$ is an *a.n.t.u.p. group* if for any artinian and narrow subsets A, B of S such that $|A| + |B| > 2$ there exist at least two u.p. elements in the product AB. But it is easy to see that $(S, \leqslant)$ is an a.n.t.u.p. group if and only if S is a t.u.p. group and the order $\leqslant$ is trivial.

Bibliography

[1] J.H Alajbegovic, N.I. Dubrovin, *Noncommutative Prüfer rings,* J. Algebra **135** (1990), 165–176.

[2] D.D. Anderson, V. Camillo, *Armendariz rings and Gaussian rings,* Comm. Algebra **26** (1998), 2265–2272.

[3] S. Annin, *Associated primes over skew polynomial rings,* Comm. Algebra **30** (2002), no. 5, 2511–2528.

[4] E. P. Armendariz, *A note on extensions of Baer and P.P.-rings,* J. Austral. Math. Soc. **18** (1974), 470–473.

[5] C. Bakkari, N. Mahdou, H. Mouanis, *Prüfer-like conditions in subring retracts and applications,* Comm. Algebra **37** (2009), 47–55

[6] G. F. Birkenmeier, H. E. Heatherly, E. K. Lee, *Completely prime ideals and associated radicals,* in (eds. S. K. Jain, S. T. Rizvi) *Ring Theory (Granville, OH, 1992),* World Scientific, Singapore and River Edge, NJ, (1993), 102–129.

[7] G. F. Birkenmeier, J. K. Park, *Triangular matrix representations of ring extensions,* J. Algebra **265** (2003), no. 2, 457–477.

[8] S. Bazzoni, S. Glaz, *Gaussian properties of total rings of quotients,* J. Algebra **310** (2007), 180–193.

[9] C. Bessenrodt, H.H. Brungs, G. Törner, *Right chain rings, Part 1,* Schriftenreihe des Fachbereichs Math. **181** Duisburg Univ., 1990.

[10] J. Brewer, E. Rutter, J. Watkins, *Coherence and weak global dimension of $R[[x]]$ when R is von Neumann regular,* J. Algebra **46** (1977), 278–289.

[11] V. Camillo, P. P. Nielsen, *McCoy rings and zero-divisors,* J. Pure Appl. Algebra **212** (2008), no. 3, 599–615.

[12] W. Chen, W. Tong, *A note on skew Armendariz rings,* Comm. Algebra **33** (2005), 1137–1140.

[13] P.M. Cohn, *Free Rings and Their Relations,* Second Edition, Academic Press, London, 1985.

[14] A. W. Chatters, W. M. Xue, *On right duo p.p. rings,* Glasgow Math. J. **32** (1990), no. 2, 221–225.

[15] G.A. Elliott, P. Ribenboim, *Fields of generalized power series,* Arch. Math. **54** (1990), 365–371.

[16] S. P. Farbman, *The unique product property of groups and their amalgamated free products,* J. Algebra **178** (1995), no. 3, 962–990.

[17] M. Ferrero, R. Mazurek, A. Sant'Ana, *On right chain semigroups,* J. Algebra **292** (2005), 574–584.

[18] M. Ferrero, A. Sant'Ana, *On distributive modules and rings,* Result. Math. **44** (2003), 74–85.

[19] L. Fuchs, S. Shelah, *Kaplansky's problem on valuation rings,* Proc. Amer. Math. Soc. **105** (1989), no. 1, 25–30.

[20] R. Gilmer, *Some applications of the Hilfssatz von Dedekind Mertens,* Math. Scand. **20** (1967), 240–244.

[21] S. Glaz, *The weak dimensions of Gaussian rings,* Proc. Amer. Math. Soc. **133** (2005), 2507–2513.

[22] S. Glaz, Prüfer conditions in rings with zero-divisors. *Arithmetical properties of commutative rings and monoids,* 272–281, Lect. Notes Pure Appl. Math., **241**, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[23] N. J. Groenewald, *A note on extensions of Bear and P.P.-rings,* Publ. Inst. Math. (Beograd) (N.S.) **34(48)** (1983), 71–72.

[24] C. M. Ha, C. Huh, H. K. Kim, N. K. Kim, Y. Lee, *On a hereditary radical property relating to the reducedness,* preprint

[25] E. Hashemi, A. Moussavi, *Polynomial extensions of quasi-Baer rings,* Acta Math. Hungarica **107** (2005), no. 3, 207–224.

[26] D. Herbera, *Bézout and semihereditary power series rings,* J. Algebra **270** (2003), 150–168.

[27] G. Higman, *Ordering by divisibility in abstract algebras,* Proc. London Math. Soc. (3) **2** (1952), 326–336.

[28] Y. Hirano, *On annihilator ideals of a polynomial ring over a noncommutative ring,* J. Pure Appl. Algebra **168** (2002), 45–52.

[29] C.Y. Hong, N.K. Kim, T.K. Kwak, *Ore extensions of Baer and p.p.-rings,* J. Pure Appl. Algebra **151** (2000), 215–226.

[30] C. Y. Hong, N. K. Kim, T. K. Kwak, *On skew Armendariz rings,* Comm. Algebra **31** (2003), no. 1, 103–122.

[31] C. Huh, H. K. Kim, Y. Lee, *Questions on 2-primal rings,* Comm. Algebra **26** (1998), no. 2, 595–600.

[32] C. Huh, Y. Kim, A. Smoktunowicz, *Armendariz rings and semicommutative rings,* Comm. Algebra **30** (2002), no. 2, 751–761.

[33] A. Jategaonkar, *Skew polynomial rings over orders in Artinian rings,* J. Algebra **21** (1972), 51–59.

[34] O.A.S. Karamzadeh, A.A. Koochakpoor, *On $\aleph_0$-self-injectivity of strongly regular rings,* Comm. Algebra **27** (1999), 1501–1513.

[35] N. K. Kim, Y. Lee, *Armendariz rings and reduced rings,* J. Algebra **223** (2000), no. 2, 477–488.

[36] N. K. Kim, K. H. Lee, Y. Lee, *Power series rings satisfying a zero divisor property,* Comm. Algebra **34** (2006), no. 6, 2205–2218.

[37] J. Krempa, *Some examples of reduced rings,* Algebra Colloq. **3** (1996), 289–300.

[38] J.B. Kruskal, *The theory of well-quasi-ordering: A frequently discovered concept,* J. Combinatorial Theory (A) **13** (1972), 297–305.

[39] T.Y. Lam, *A First Course in Noncommutative Rings,* Springer-Verlag, New York, 1991.

[40] T.Y. Lam, *Exercises in classical ring theory,* Springer, New York, 1995.

[41] T.Y. Lam, *Lectures on Modules and Rings,* Springer, New York 1998

[42] T.-K. Lee, T.-L. Wong, *On Armendariz rings,* Houston J. Math. **29** (2003), no. 3, 583–593.

[43] T.-K. Lee, Y. Zhou, *A unified approach to the Armendariz property of polynomial rings and power series rings,* Colloq. Math. **113** (2008), no. 1, 151–168.

[44] Z. Liu, *Special properties of rings of generalized power series,* Comm. Algebra **32** (2004), no. 8, 3215–3226.

[45] Z. Liu, *Armendariz rings relative to a monoid,* Comm. Algebra **33** (2005), no. 3, 649–661.

[46] J.C McConnell, J.C. Robson, *Noncommutative Noetherian Rings,* Wiley, New York, 1987.

[47] G. Marks, *Direct product and power series formations over 2-primal rings,* in (eds. S. K. Jain, S. T. Rizvi) *Advances in Ring Theory,* Trends in Math., Birkhäuser, Boston (1997), 239–245.

[48] G. Marks, *Skew polynomial rings over 2-primal rings,* Comm. Algebra **27** (1999), no. 9, 4411–4423.

[49] G. Marks, *Reversible and symmetric rings,* J. Pure Appl. Algebra **174** (2002), no. 3, 311–318.

[50] G. Marks, *A taxonomy of 2-primal rings,* J. Algebra **266** (2003), no. 2, 494–520.

[51] G. Marks, *Duo rings and Ore extensions,* J. Algebra **280** (2004), 463–471.

[52] G. Marks, R. Mazurek, M. Ziembowski, *A new class of unique product monoids with applications to ring theory,* Semigroup Forum **78** (2009), 210–225.

[53] J. Matczuk, *A characterization of σ-rigid rings,* Comm. Algebra **32** (2004), no. 11, 4333–4336.

[54] G. Marks, R. Mazurek, M. Ziembowski, *A unified approach to various generalizations of Armendariz rings,* Bull. Austral. Math. Soc. **81** (2010), 361–397.

[55] R. Mazurek, M. Ziembowski, *On Bézout and distributive generalized power series rings,* J. Algebra **306** (2006), 387–401.

[56] R. Mazurek, M. Ziembowski, *Uniserial rings of skew generalized power series,* J. Algebra **318** (2007), 737–764.

[57] R. Mazurek, M. Ziembowski, *On von Neumann regular rings of skew generalized power series,* Comm. Algebra **36** (2008), no. 5, 1855–1868.

[58] R. Mazurek, M. Ziembowski, *Duo, Bézout and distributive rings of skew power series,* Publ. Mat. **53** (2009), no. 2, 257–271.

[59] R. Mazurek, M. Ziembowski, *Weak dimension and right distributivity of skew generalized power series rings,* to appear in Journal of the Mathematical Society of Japan.

[60] R. Mazurek, M. Ziembowski, *Right Gaussian rings and skew power series rings,* submitted.

[61] A. Moussavi, E. Hashemi, *On (α, δ)-skew Armendariz rings,* J. Korean Math. Soc. **42** (2005), no. 2, 353–363.

[62] A. R. Nasr-Isfahani, A. Moussavi, *On classical quotient rings of skew Armendariz rings,* Int. J. Math. Math. Sci. **2007** (2007), Article ID 61549, 7 pages.

[63] J. Okniński, *Semigroup algebras,* Monographs and Textbooks in Pure and Applied Mathematics, **138**, Marcel Dekker, Inc., New York, 1991.

[64] D. S. Passman, *The Algebraic Structure of Group Rings,* reprint of the 1977 original, Robert E. Krieger Publishing Co., Inc., Melbourne, FL (1985).

[65] E. R. Puczyłowski, *Questions related to Koethe's nil ideal problem,* Algebra and its applications, 269–283, Contemp. Math., **419**, Amer. Math. Soc., Providence, RI, 2006.

[66] G. Puninski, *Projective modules over the endomorphism ring of a biuniform module,* J. Pure Appl. Algebra **188** (2004), no. 1-3, 227–246.

[67] M. B. Rege, S. Chhawchharia, *Armendariz rings,* Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 1, 14–17.

[68] P. Ribenboim, *Noetherian rings of generalized power series,* J. Pure Appl. Algebra **79** (1992), no. 3, 293–312.

[69] P. Ribenboim, *Semisimple rings and von Neumann regular rings of generalized power series,* J. Algebra **198** (1997), 327–338.

[70] J.C. ROBSON, *Non-commutative Dedekind rings,* J. Algebra **9** (1968), 249–265.

[71] C. S. NAGORE, M. SATYANARAYANA, *On naturally ordered semigroups,* Semigroup Forum **18** (1979), no. 2, 95–103.

[72] W. STEPHENSON, *Modules whose lattice of submodules is distributive,* Proc. London Math. Soc. **28** (1974), 291–310.

[73] A. STROJNOWSKI, *A note on u.p. groups,* Comm. Algebra **8** (1980), no. 3, 231–234.

[74] H. TSANG, *Gauss's Lemma,* Ph.D. Thesis, University of Chicago 1965.

[75] A.A. TUGANBAEV, *Integrally closed Noetherian rings,* Russ. Math. Surv. **36** (1981), 183–184

[76] A.A. TUGANBAEV, *Rings with flat right ideals and distributive rings,* Math. Notes **2** (1985), 631–636.

[77] A.A. TUGANBAEV, *Rings of series, and weak global dimension* (Russian), Izv. Vyssh. Uchebn. Zaved. Mat. 11 (1987), 70-78. (English translation: Soviet Math. (Iz. VUZ) 31 (1987), no. 11, 90-101.)

[78] A.A. TUGANBAEV, *Distributive rings and modules* (Russian), Mat. Zametki 47 (1990), 115–123. (English translation: Math. Notes 47 (1990), no. 1-2, 199–206.)

[79] A.A. TUGANBAEV, *The structure of distributive rings (Russian),* Mat. Sb. **193** (2002), no. 5, 113–128; translation in Sb. Math. 193 (2002), no. 5-6, 745–760.

[80] A. TUGANBAEV, *Distributive modules and related topics*, Algebra, Logic and Applications, Vol. 12, Gordon and Breach Science Publishers, Amsterdam, 1999.

[81] A.A. TUGANBAEV, *Distributive modules and rings* In: Mikhalev, A.V., Pilz, G.F. (Eds.), The Concise Handbook of Algebra. Kluwer Acad. Publ., Dordrecht, 2002, pp. 280–284.

[82] D.A. TUGANBAEV, *Laurent series rings and pseudo-diferential operator rings,* Journal of Mathematical Sciences **128** (2005), 2843–2893.

[83] X.-F. YAN, *Special properties of rings of twisted generalized power series,* Xibei Shifan Daxue Xuebao Ziran Kexue Ban **43** (2007), no. 2, 20–23.

[84] Y. ZHOU, *Noncommutative Prüfer rings and some generalizations,* Comm. Algebra **20** (1992), 2609–2633.

Index